IM2010: Operations Research
Inventory Models
(Chapters 15 and 16)

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Road map

- Introduction.
- The EOQ model.
- Variants of the EOQ model.
- The newsvendor model.
What are inventory?

▶ For almost all firms producing or purchasing products to sell, they need inventory.

▶ Why inventory?
  ▶ If each batch of production or procurement requires some fixed costs, we will increase the batch size to save money.
  ▶ If demand is uncertain, inventory provides a buffer for supply-demand mismatch.

▶ Key questions in the manufacturing and retailing industries regarding inventory include:
  ▶ When to do replenishment?
  ▶ How much to replenish?
  ▶ From which suppliers?

▶ In this session, we introduce fundamental OR models that make the optimal inventory decisions.
An LP-based inventory model

- We have seen the following inventory model:
  - We have $T$ periods with different demands.
  - In each period, we first produce and then sell.
  - Unsold products become ending inventories.
  - We want to minimize the total cost.
  - In period $t$, $C_t$ is the unit production cost, $D_t$ is the unit production quantity, and $H$ is the unit holding cost per period.

- The formulation is

\[
\min \sum_{t=1}^{T} (C_t x_t + H y_t)
\]

s.t.

\[
y_{t-1} + x_t - D_t = y_t \quad \forall t = 1, \ldots, T
\]

\[
y_0 = 0
\]

\[
x_t, y_t \geq 0 \quad \forall t = 1, \ldots, T.
\]
Categories of inventory models

- The previous model is an example of a periodic review system with deterministic demands.
  - Replenishment can occur at most once per “period”.
  - All future demands are perfectly predicted.
- In a continuous review system, one may replenish at any time point.
- When we are facing inventory decisions, in general there are four types of inventory systems:

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<th>Review time</th>
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Two NLP-based inventory models

- We will introduce two NLP-based inventory models:
  - The **economic order quantity** (EOQ) model.
  - The **newsvendor** model.

- They are basic, fundamental, widely applied.
- They are the foundations of most advanced inventory models.
- They are the applications of (single-variate) NLP.
- How to categorize them?

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- Variants of the EOQ model.
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Motivating example

- IM Airline uses 500 taillights per year. It purchases these taillights from a manufacturer at a unit price $500.
- Taillights are consumed at a constant rate throughout a year.
- Whenever IM Airline places an order, an ordering cost of $5 is incurred regardless of the order quantity.
- The holding cost is 2 cents per taillight per month.
- IM Airline wants to minimize the total cost, which is the sum of ordering, purchasing, and holding costs.
- How much to order? When to order?
  - What is the benefit of having a small or large order?
The EOQ model

- IM Airline’s question may be answered with the economic order quantity (EOQ) model.
- In this model, we try to find the order quantity that is the most economic.
  - In particular, we want to find a balance between the ordering cost and holding cost.
- Technically, we will formulate an NLP whose optimal solution is the optimal order quantity.
Assumptions of the EOQ model

- Demand is deterministic and occurs at a constant rate.
- Regardless the order quantity, a fixed ordering cost is incurred.
- No shortage is allowed.
- The ordering lead time is zero.
- The inventory holding cost is constant.
Parameters and the decision variable

- **Parameters:**

  \[
  D = \text{annual demand (units)},
  \]
  \[
  K = \text{unit ordering cost ($)},
  \]
  \[
  h = \text{unit holding cost per year ($), and}
  \]
  \[
  p = \text{unit purchasing cost ($)}.
  \]

- **Decision variable:**

  \[
  q = \text{order quantity per order (units)}.
  \]

- **Objective:** Minimizing annual total cost.

- For all our calculations, we will use **one year** as our time unit. Therefore, \( D \) can be treated as the demand **rate**.
Inventory level by time

- To formulate the problem, we need to understand how the inventory level is affected by our decision.
- Based on the EOQ assumptions, we will always place an order when the inventory level is zero.
- As inventory is consumed at a constant rate, the inventory level vary in time in the following way:
Annual costs

- Annual holding cost = \( h\left(\frac{q}{2}\right) = \frac{hq}{2} \).
  - For a time period, the holding cost is \( h \) times the area under the curve over that time period.
  - For one year, the length of the time period is 1 and the inventory level is \( \frac{q}{2} \) in average.

- Annual purchasing cost = \( pD \).
  - We need to buy \( D \) units regardless the order quantity.

- Annual ordering cost = \( K\left(\frac{D}{q}\right) = \frac{KD}{q} \).
  - The number of orders in a year is \( \frac{D}{q} \).

- Annual total cost = \( TC(q) = \frac{KD}{q} + pD + \frac{hq}{2} \).
Nonlinear optimization

- The NLP for optimizing the ordering decision is

$$\min_{q \geq 0} TC(q) = \frac{KD}{q} + pD + \frac{hq}{2}.$$  

- We have

$$TC'(q) = -\frac{KD}{q^2} + \frac{h}{2},$$  
and

$$TC''(q) = \frac{2KD}{q^3} > 0.$$  

Therefore, $TC(q)$ is convex in $q$.  

Optimizing the order quantity

- Let \( q^* \) be the quantity satisfying the FOC:

\[
TC'(q^*) = -\frac{KD}{(q^*)^2} + \frac{h}{2} = 0 \quad \Rightarrow \quad q^* = \sqrt{\frac{2KD}{h}}.
\]

- As this quantity is feasible, it is optimal.
- The resulting annual holding and ordering cost is \( \sqrt{2KDh} \).
- The optimal order quantity \( q^* \) is the EOQ. It is:
  - Increasing in the ordering cost \( K \).
  - Increasing in the annual demand \( D \).
  - Decreasing in the holding cost \( h \).
  - Unaffected by the purchasing cost \( p \).

Why?
Example

- IM Airline uses 500 taillights per year.
- The ordering cost is $5 per order.
- The holding cost is 2 cents per unit per month.
- Taillights are consumed at a constant rate.
- No shortage is allowed.
- Questions:
  - What is the EOQ?
  - How many orders to place in each year?
  - What is the order cycle time (time between two orders)?
Example: the optimal solution

- The EOQ is

\[ q^* = \sqrt{\frac{2(5)(500)}{(0.24)}} \approx \sqrt{20833.33} \approx 144.34 \text{ unit}. \]

- Make sure that time units are consistent!
- The average number of orders in a year is \( \frac{500}{q^*} \approx 3.464 \) orders.
- The order cycle time is

\[ T^* = \frac{1}{3.464} \approx 0.289 \text{ year} \approx 3.464 \text{ months}. \]

- The number of orders in a year and the order cycle time are the same! Is it a coincidence?
Example: cost analysis

- The annual holding cost is $\frac{hq_*}{2} \approx 17.32$.
- The annual ordering cost is $\frac{KD}{q_*} \approx 17.32$.
- The two costs are identical! Is it a coincidence?
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- **Variants of the EOQ model.**
- The newsvendor model.
Nonzero lead time

- What if there is an *ordering lead time* $L > 0$?
  - This means after we place an order, we will receive the product after $L$ year.

- In this case, we want to calculate the *reorder point*, which is the inventory level at which an order should be placed.
Reorder points

- When to order?
- Let $R$ be the reorder point. We want to calculate $R$ such that we receive products exactly when we have no inventory.
- If $L \leq T^*$:
  \[ R = LD. \]
  - $T^*$ is the order cycle time.
  - $L$ must be measured in years!
- If $L \geq T^*$:
  \[ R = D(L - kT^*) \]
  for some $k \in \mathbb{N}$ such that $0 \leq L - kT^* \leq T^*$. 
Economic production quantity (EPQ)

- When products are produced rather than purchased, typical they are “received” at a continuous rate.
- The model that finds the optimal production lot size is called the economic production quantity (EPQ) model.
- Under the assumption that the product is produced at a constant rate of $r$ units per year, what lot size minimizes the total cost?
Economic production quantity (EPQ)

- The inventory level now looks like:
Economic production quantity

- Suppose we choose $q$ as our production lot size.
- In the production time, the demand increases at the rate $r - D$.
  - While we produce at the rate $r$, we also consume at the rate $D$.
- The length of the production time is $\frac{q}{r}$ year. Why?
- So the maximum inventory level (achieved at the end of a production period) is $(r - D)\frac{q}{r}$.
Economic production quantity

- Still, the amount we produce in a lot will be depleted in \( \frac{q}{D} \) year.
Economic production quantity

- The annual holding cost now becomes
  \[ h \left[ \frac{q(r - D)}{2r} \right]. \]

- The average inventory level is \( \frac{1}{2}[(r - D)\frac{q}{r}] \).
- The annual setup cost is still \( K(\frac{D}{q}) \).
- The purchasing cost still does not affect the decision.
- The total holding and setup cost is:
  \[ \frac{hq(r - D)}{2r} + \frac{KD}{q}. \]
Economic production quantity

- Note that this is the same as the EOQ model

\[ \frac{hq}{2} + \frac{KD}{q}. \]

if we let \( h(\frac{r-D}{r}) = h(1 - \frac{D}{r}) \) be the effective holding cost.

- The optimal production lot size is thus

\[ q^* = \sqrt{\frac{2KD}{h(1 - \frac{D}{r})}}, \]

which is the EPQ we desire.
Example

- IM Auto needs to produce 10000 cars per year.
- Each car requires $2000 to produce.
- Each run requires $200 to set up.
- Annual holding cost rate is 25%:
  - The holding cost per car per year is $\frac{2000}{4} = 500$.
- The production rate is 25000 cars per year.
- What is the EPQ and optimal cycle time?
Example

- The EPQ is
  \[
  \sqrt{\frac{2(200)(10000)}{500(1 - \frac{10000}{25000})}} = 115.47 \text{ cars.}
  \]

- The optimal cycle time is
  \[
  \frac{1}{\frac{10000}{115.47}} \approx 0.012 \text{ year} \approx 4.21 \text{ days.}
  \]

- Will the annual holding cost and annual setup cost still be identical? Why?
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The newsvendor model

- In some situations, people sell perishable products.
  - Perishable products will become valueless after the selling season is end.
  - E.g., newspapers become valueless after one day.
  - High-tech goods are valueless once the next generation is offered.
  - Fashion goods become valueless when they become out of fashion.

- For perishable products, sometimes the seller only have one chance for replenishment.
  - E.g., a small newspaper seller can order only once and obtain those newspapers only at the morning of each day.

- Often sellers of perishable products face uncertain demands.

- The question is how many products one should prepare for the entire selling season.
Newsvendor model

Let $D$ be the uncertain demand.
Let $F$ and $f$ be the cdf and pdf of $D$ (assuming $D$ is continuous).
Let $c_o$ be the **overage cost** and $c_u$ be the **underage cost**.
  - They are also called overstocking and understocking costs.
  - They are the costs for prepare too many or too few products.
We want to find an order quantity that minimize the expected total overage and underage costs.
  - As demands are uncertain, we try to minimize the **expectation** of the random cost.
Components of overage and underage costs

- Components of overage and underage costs may include:
  - Sales revenue $r$ for each unit sold.
  - Purchasing cost $c$ for each unit purchased.
  - Salvage value $a$ for each unit unsold.
  - Disposal fee $p$ for each unit unsold.
  - Shortage cost $s$ for each unit of shortage.
- With these quantities, we have
  - The overage cost $c_o = c + p - a$.
  - The underage cost $c_u = r - c + s$. 
Formulation of the newsvendor problem

- Let \( q \) be the order quantity (inventory level).
- Let \( d \) be the realization of demand.
  - \( D \) is a random variable and \( d \) is a realized value of \( D \).
- Then the cost is

\[
c(q, d) = \begin{cases} 
  c_o(q - d) & \text{if } q \geq d \\
  c_u(d - q) & \text{if } q < d.
\end{cases}
\]

- Now, the expected total cost is

\[
c(q, D) = \mathbb{E} \left[ c_o(q - d)^+ + c_u(d - q)^+ \right],
\]

where \( x^+ = \max(x, 0) \).
Convexity of the cost function

We want to find a quantity $q$ that solves

$$\min_{q \geq 0} \mathbb{E} \left[c_o(q - d)^+ + c_u(d - q)^+\right].$$

By assuming that $D$ is continuous, the cost function $c(q, D)$ is

\[
\begin{align*}
\int_0^q \left[c_o(q - x) + c_u \cdot 0\right] f(x)dx + \int_q^\infty \left[c_o \cdot 0 + c_u(x - q)\right] f(x)dx \\
= c_o \int_0^q (q - x)f(x)dx + c_u \int_q^\infty (x - q)f(x)dx \\
= c_o \left[q \int_0^q f(x)dx - \int_0^q xf(x)dx\right] + c_u \left[\int_q^\infty xf(x)dx - q \int_q^\infty f(x)dx\right] \\
= c_o \left[qF(q) - \int_0^q xf(x)dx\right] + c_u \left[\int_q^\infty xf(x)dx - q(1 - F(q))\right].
\end{align*}
\]
Convexity of the cost function

- The first-order derivative of $c(q, D)$ is
  
  $$c'(q, D) = c_o [F(q) + qf(q) - qf(q)] + c_u [-qf(q) - (1 - F(q)) + qf(q)]$$
  
  $$= c_o [F(q)] - c_u [1 - F(q)].$$

- The second-order derivative of $c(q, D)$ is
  
  $$c''(q, D) = c_of(q) - c_uf(q) = f(q)(c_u + c_0) > 0.$$  

- So $c(q, D)$ is convex in $q$. 
Optimizing the order quantity

Let $q^*$ be the order quantity that satisfies the FOC, we have

$$c_o F(q^*) - c_u (1 - F(q^*)) = 0$$

$$\Rightarrow F(q^*) = \frac{c_u}{c_o + c_u} \quad \text{or} \quad 1 - F(q^*) = \frac{c_o}{c_o + c_u}.$$

Such $q^*$ must be positive (for regular demand distributions). So $q^*$ is optimal.

Note that to minimize the expected total cost, the seller should intentionally create some shortage!

The optimal probability of having a shortage is $\frac{c_o}{c_o + c_u}$. 

\[ 
\]
Determinants of the optimal quantity

- The probability of having a shortage, \(1 - F(q)\), is decreasing in \(q\).
- The optimal quantity \(q^*\) is:
  - Decreasing in \(c_o\): When \(c_o\) increases, the optimal quantity moves from \(q^*\) to \(q_1\).
  - Increasing in \(c_u\): When \(c_u\) increases, the optimal quantity moves from \(q^*\) to \(q_2\).
Example 1

- Suppose for a newspaper:
  - The unit purchasing cost is $5.
  - The unit retail price is $15.
  - The demand is uniformly distributed between 20 to 50.
- Overage cost $c_o = 5$ and underage cost $c_u = 15 - 5 = 10$.
- The optimal order quantity $q^*$ satisfies

$$1 - F(q^*) = \left(1 - \frac{q^* - 20}{50 - 20}\right) = \frac{5}{5 + 10} \Rightarrow \frac{50 - q^*}{30} = \frac{1}{3},$$

which implies $q^* = 40$.
- If the unit purchasing cost increases to $6$, we need $\frac{50 - q^{**}}{30} = \frac{6}{15}$ and thus $q^{**} = 36$.
  - As the purchasing cost increases, we dislike overstocking more. Therefore, we stock less.
Example 2

▶ Suppose for one kind of apple:
  ▶ The unit purchasing cost is $15, the unit retail price is $21, and the unit salvage value is $1.
  ▶ The demand $D \sim \text{ND}(90, 20)$, i.e., $D$ is normally distributed with mean 90 and standard deviation 20.
  ▶ Overage cost $c_o = 15 - 1 = 14$ and underage cost $c_u = 21 - 15 = 6$.
▶ The optimal order quantity $q^*$ satisfies

\[
\Pr(D < q^*) = \frac{6}{14 + 6} \implies \Pr\left(Z < \frac{q^* - 90}{20}\right) = 0.3,
\]

where $Z \sim \text{ND}(0, 1)$.
▶ By looking at a standard normal probability table or using any Statistical software, we find $\Pr(Z < -0.5244) = 0.3$, which implies $\frac{q^* - 90}{20} = -0.5244$ and thus $q^* = 79.512$.
▶ As the purchasing cost is so high, we want to reject more than half of the consumers!