IM 7011: Information Economics

Overview and preliminaries
Lecture 1.1: Overview

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Welcome!

- This is **Information Economics**, NOT **Information Economy**.
    - This is not a course talking about how to design and sell information goods, information systems, social networks, and high-tech products.
- This is an economics course focusing on the issue of information. This is **economics of information**.
- In different business environments:
    - How people behave with different information?
    - What is the value of information?
    - What information to acquire?
    - Is knowing more always better?
- In this course, we focus on **information asymmetry**.

Information asymmetry

- The world is full of asymmetric information:
  - A consumer does not know a retailer's procurement cost.
  - A consumer does not know a product's quality.
  - A retailer does not know a consumer's valuation.
  - An instructor does not know how hard a student works.
- As information asymmetry results in inefficiency, we want to:
  - Analyze its impact. If possible, quantify it.
  - Decide whether it introduces driving forces for some phenomena.
  - Find a way to deal with it if it cannot be eliminated.
- This field is definitely fascinating. However:
  - We need to have some "**weapons**" to explore the world!

Before you enroll...

- Prerequisites:
  - Calculus.
  - Convex optimization.
  - Probability.
  - Game theory.
- Language: **"All" English**.
  - All materials are in English.
  - Students should try their best to speak English in class. But when it really helps, one may speck Chinese.
  - The instructor will speak Chinese in office hour unless a student prefers English.
  - The instructor will speak Chinese in lectures when it helps.

The instructor

- Ling-Chieh Kung.
  - Second-year assistant professor.
  - Office: Room 413, Management Building II.
  - Office hour: **9:10am-11:10am, Thursday** or by appointment.
  - E-mail: lckung@ntu.edu.tw.
- There is no teaching assistant for this course.

Related information

- Classroom: Room 204, Management Building II.
- Lecture time: 9:10am-12:10pm, Monday.
- Main references:
  - *Contract Theory* by P. Bolton and M. Dewatripont.
  - Around ten academic papers.
- References:
  - *Game Theory for Applied Economists* by R. Gibbons.
  - *The Theory of Incentives: The Principal-agent Model* by J.-J. Laffont and D. Martimort.
  - *Information Rules: A Strategic Guide to the Network Economy* by C. Shapiro and H. Varian.
  - *Auction Theory* by V. Krishna.

"Flipped classroom"

- Lectures in **videos**, then discussions in classes.
- Before each Monday, the instructor uploads a video of lectures.
  - Ideally, the video will be no longer than one and a half hour.
  - Students must watch the video by themselves before that Monday.
- During the lecture, we do three things:
  - Discussing the lecture materials (0.5 to 1 hour).
  - Solving **class problems** (1 to 2 hours).
  - Further discussions (0.5 to 1 hour).
- After the lecture, students also need to do homework.

Teams

- Students form **teams** to do class problems and homework.
- Each team has **three** students.
  - Unless a special approval is obtained.
- Students may change teammates from homework to homework.
- Once some students form a team for one homework, they will be **in the same team** for class problems until the submission of the next homework.
- All students get the same grades for each homework and class problem.

Homework and class problems

- Homework:
  - Homework will be assigned roughly once per two weeks.
  - For each homework, each team needs to submit only one paper.
  - Please put a **hard copy** of your work into my **mailbox** on the first floor of the Management Building II by the due time.
  - No submission in class. No late submission.
  - The lowest one homework grade will be dropped (i.e., you may skip one homework if you want).
- Class problems:
  - For each problem assigned by the instructor in class, students discuss in teams for around 10 minutes.
  - At least **one team** then demonstrate their answer to the class (in **English**) to get grades for class problems.
  - Sometimes teams may volunteer; sometimes the instructor determines who to answer.

Class participation and office hour

- Class participation:
  - We do not require one to attend all the lectures.
  - However, those who participate in class discussions get rewarded.
  - Class problems also count for grades.
  - Missing a class makes it impossible for you and less possible for your teammates to get this part of grades.
- Office hour:
  - Come to discuss any question (or just chat) with me!
  - If the regular time does not work for you, just send me an e-mail.
  - My “open-door” policy.

Projects and exams

- Project:
  - Please form a new team of at most n students, where the value of n will be determined according to the class size.
  - Each team will write a research proposal for a self-selected topic, make a 30-minute presentation, and submit a report.
  - All team members must be in class for the team to present.
- Two exams:
  - In-class and open whatever you have (including all kinds of electronic devices).
  - No discussion is allowed. Cheating will result in severe penalty.
  - The final exam is comprehensive.

Grading

- Homework: 20%.
- Projects: 20%.
- Class problems: 15%.
- Class participation: 5%.
- Two Exams: 40%:
  - Plan 1: midterm 20% and final 20%.
  - Plan 2: midterm 15% and final 25%.
- The final letter grades will be given according to the following conversion rule:

Letter	Range	Letter	Range	Letter	Range
A+	$[90, 100]$	B+	$[77, 80)$	C+	$[67, 70)$
A	$[85, 90)$	B	$[73, 77)$	C	$[63, 67)$
A−	$[80, 85)$	B−	$[70, 73)$	C−	$[60, 63)$

Important dates and tentative plan

- Important dates:
  - Week 5 (2013/10/7): No class because the instructor is in a conference.
  - Week 9 (2013/11/4): Midterm exam.
  - Weeks 16 and 17 (2013/12/23 and 30): Project presentation.
  - Week 18 (2014/1/6): Final exam.
- Tentative plan:
  - Review of optimization and game theory.
  - Contracting without information asymmetry.
  - Hidden information: screening (Ch. 2 of Contract Theory).
  - Hidden information: signaling (Ch. 3 of Contract Theory).
  - Hidden action: moral hazard (Ch. 4 of Contract Theory).
  - Advanced topics (Ch. 6 and 7 of Contract Theory).

Online resources

- CEIBA.
  - Viewing your grades.
  - Receiving announcements.
- http://www.ntu.edu.tw/~lckung/courses/IEFa13/.
  - Downloading course materials.
- The bulletin board "NTUIM-lckung" on PTT.
  - Discussions.
- YouTube:
  - Watching lecture videos.

IM 7011: Information Economics

Overview and preliminaries
Lecture 1.2: Convexity, Optimization, and Probability

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Road map

- Convexity.
- Optimization problems.
- Distributions and expectations.

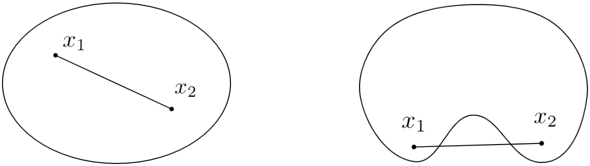
Convex sets

Definition 1 (Convex sets)

A set F is **convex** *if*

$$\lambda x_1 + (1-\lambda)x_2 \in F$$

for all $\lambda \in [0,1]$ and $x_1, x_2 \in F$.



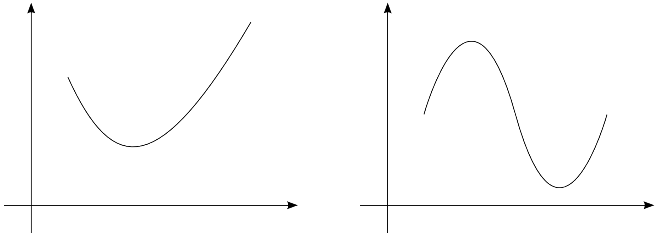
Convex functions

Definition 2 (Convex functions)

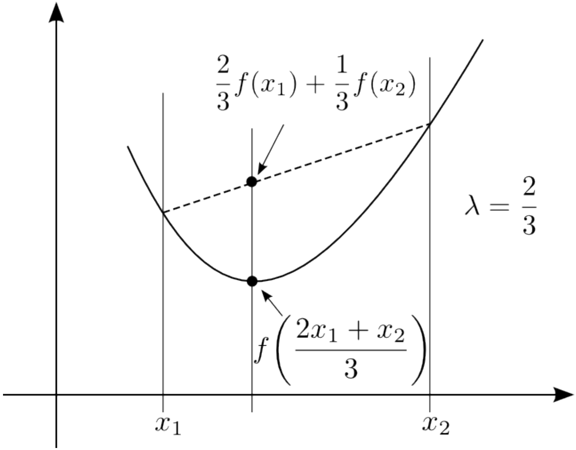
For a convex domain F, a function $f(\cdot)$ is **convex** *over F if*

$$f\Big(\lambda x_1 + (1-\lambda)x_2\Big) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

for all $\lambda \in [0,1]$ and $x_1, x_2 \in F$.



Convex functions



Some examples

- Convex sets?
  - $X_1 = [10, 20]$.
  - $X_2 = (10, 20)$.
  - $X_3 = \mathbb{N}$.
  - $X_4 = \mathbb{R}$.
  - $X_5 = \{(x, y) | x^2 + y^2 \leq 4\}$.
  - $X_6 = \{(x, y) | x^2 + y^2 \geq 4\}$.
- Convex functions?
  - $f_1(x) = x + 2, x \in \mathbb{R}$.
  - $f_2(x) = x^2 + 2, x \in \mathbb{R}$.
  - $f_3(x) = \sin(x), x \in [0, 2\pi]$.
  - $f_4(x) = \sin(x), x \in [\pi, 2\pi]$.
  - $f_5(x) = \log(x), x \in (0, \infty)$.
  - $f_6(x, y) = x^2 + y^2, (x, y) \in \mathbb{R}^2$.

Strictly convex and concave functions

Definition 3 (Strictly convex functions)

For a convex domain F, a function $f(\cdot)$ is **strictly convex** *over F if*

$$f\Big(\lambda x_1 + (1-\lambda)x_2\Big) < \lambda f(x_1) + (1-\lambda) f(x_2)$$

for all $\lambda \in (0,1)$ and $x_1, x_2 \in F$ such that $x_1 \neq x_2$.

Definition 4 ((Strictly) concave functions)

For a convex domain F, a function $f(\cdot)$ is **(strictly) concave** *over F if $-f(\cdot)$ is (strictly) convex.*

Derivatives of convex functions

Proposition 1

A single-variate twice-differentiable function $f(\cdot)$ is **convex** *over an interval $[a, b]$ if and only if*

$$f''(x) \geq 0 \quad \forall x \in [a, b].$$

Proposition 2

A single-variate twice-differentiable function $f(\cdot)$ is **strictly convex** *over an interval $[a, b]$ if and only if*

$$f''(x) > 0 \quad \forall x \in [a, b].$$

Derivatives of concave functions

Proposition 3

A single-variate twice-differentiable function $f(\cdot)$ is **concave** *over an interval $[a, b]$ if and only if*

$$f''(x) \leq 0 \quad \forall x \in [a, b].$$

Proposition 4

A single-variate twice-differentiable function $f(\cdot)$ is **strictly concave** *over an interval $[a, b]$ if and only if*

$$f''(x) < 0 \quad \forall x \in [a, b].$$

Road map

- Convexity.
- **Optimization problems**.
- Distributions and expectations.

Optimization problems

- In an optimization problem, there are:
  - **Decision variables**.
  - **The objective function**.
  - **Constraints**.
- Consider the well-known *knapsack* problem:
  - I have n items.
  - The value and weight of item i are p_i and w_i (in kg), respectively.
  - I can carry at most B kg.
  - I want to maximize the total value of items I carry.

Formulation

- Decision variables: Let

$$x_i = \begin{cases} 1 & \text{if I carry item } i \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., n.$$

- The objective function:

$$\max \sum_{i=1}^{n} p_i x_i.$$

- Capacity constraint:

$$\sum_{i=1}^{n} w_i x_i \leq B.$$

- Binary constraint:

$$x_i \in \{0, 1\} \quad \forall i = 1, ..., n.$$

Formulation

- The complete formulation:

$$\begin{aligned} z^* = \max \quad & \sum_{i=1}^{n} p_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} w_i x_i \leq B \\ & x_i \in \{0, 1\} \quad \forall i = 1, ..., n. \end{aligned}$$

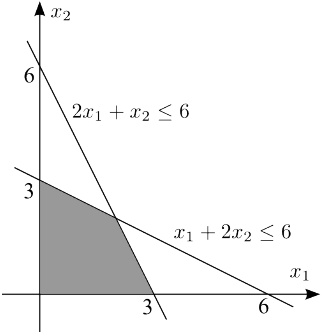
- Suppose $n = 3$, $p = (15, 20, 25)$, $w = (5, 4, 7)$, $B = 9$.
  - The **feasible region** (the set of all **feasible solutions**) is $\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0)\}$.
  - Solutions $(1, 0, 1)$, $(0, 1, 1)$, and $(1, 1, 1)$ are **infeasible**.
  - An **optimal solution** is $x^* = (1, 1, 0)$. It happens to be **unique**.
  - The optimal objective value is $z^* = 35$.
- For this course, most problems will contain only continuous variables.

Linear programming

- Consider the problem

$$
\begin{aligned}
z^* = \max \quad & 2x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 6 \\
& 2x_1 + x_2 \leq 6 \\
& x_i \geq 0 \quad \forall i = 1, 2.
\end{aligned}
$$

  - The feasible region is the shaded area.
  - There are multiple optimal solutions (where?).
  - There is still a unique optimal objective value $z^* = 6$.
- An optimization problem is a **linear program** (LP) if the objective function and constraints are all linear.



Nonlinear programming

- An optimization problem is a **convex program** if in it we maximize a concave function over a convex feasible region.
- Consider the convex program

$$\begin{aligned} z^* = \max \quad & \log_2 x_1 + \log_2 x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 16 \\ & x_1 + x_2 \geq 1. \end{aligned}$$

  - What is the feasible region?
  - What is an optimal solution? Is it unique?
  - What is the value of z^*?
- All convex programs can be solved efficiently.
- A problem is a **nonlinear program** if it is not a linear program.
- It may not be possible to solve a nonconvex program efficiently.

Infeasible and unbounded problems

- Not all problems have an optimal solution.

Definition 5 (Infeasible problems)

A problem is **infeasible** *if there is no feasible solution.*

  - E.g., $\max\{x^2 | x \leq 2, x \geq 3\}$.

Definition 6 (Unbounded problems)

A problem is **unbounded** *if given any feasible solution, there is another feasible solution that is better.*

  - E.g., $\max\{e^x | x \geq 3\}$.
  - How about $\min\{\sin x | x \geq 0\}$?

- A problem may be infeasible, unbounded, or having an optimal solution (may or may not be unique).

Set of optimal solutions

- The **set of optimal solutions** of a problem $\max\{f(x)|x \in X\}$ is
$$\operatorname{argmax}\{f(x)|x \in X\}.$$
- Let $X = \left\{x_1 + 2x_2 \leq 6, 2x_1 + x_2 \leq 6, x \in \mathbb{Z}_+^2\right\}$. We have
$$12 = \max\left\{4x_1 + 2x_2 \middle| x \in X\right\}$$
and
$$\left\{(2,2),(3,0)\right\} = \operatorname{argmax}\left\{2x_1 + x_2 \middle| x \in X\right\}.$$
- If x^* is an optimal solution of $\max\{f(x)|x \in X\}$, we should write $x^* \in \operatorname{argmax}\{f(x)|x \in X\}$, NOT $x^* = \operatorname{argmax}\{f(x)|x \in X\}$!

Road map

- Convexity.
- Optimization problems.
- **Distributions and expectations**.

Random variables

- The value of a **random variable** is unknown before it is **realized**.
- A random variable may be discrete, continuous, or mixed.
  - A **discrete** one models a quantity that is typically **counted**.
  - A **continuous** one models a quantity that is typically **measured**.
  - A mixed one has one part discrete and the other part continuous.

Discrete random variables

- A discrete random variable is described by its **probability mass function** (pmf).
  - Let Y be the outcome of tossing a fair dice. What is the pmf of Y?
  - Let Z be the sum of two fair dices. What is the pmf of Z?
- Let X be a discrete random variable. Its pmf, $p_X(\cdot)$, is a function mapping a possible realization to a real values between 0 and 1 (which is the **probability**).
  - $p_X : S \to [0, 1]$, where S is the sample space of X.
  - What is $p_Y(3) = \Pr(Y = 3)$? What is $p_Z(3) = \Pr(Z = 3)$?

Continuous random variables

- A continuous random variable is described by its **probability density function** (pdf).
  - Let Y be uniformly distributed with lower bound a and upper bound b. The pdf of Y is

$$f_Y(y) = \frac{1}{b-a} \quad \forall y \in [a, b].$$

  - Let Z be exponentially distributed with rate λ. The pdf of Z is

$$f_Z(z) = \lambda e^{-\lambda z} \quad \forall z \in [0, \infty).$$

- Let X be a continuous random variable. Its pdf, $f_X(\cdot)$, is now a function mapping a possible realization to a nonnegative real value.
  - $f : S \to [0, \infty)$, where S is the sample space of X.
- This value is NOT a probability!
  - What is $f_Y(3)$? Is it $\Pr(Y = 3)$?

Continuous random variables

- For a continuous random variable X, the probability for X to be **equal to a value** is always 0.
- Only the probability for X to be **within a range** can be measured.
  - Let $Y \sim f_Y$ where $f_Y(y) = \frac{1}{4}$ for $y \in [0, 4]$. What is $\Pr\Big(Y \in [3, 4]\Big)$?
  - Let $Z \sim f_Z$ where $f_Z(z) = 2e^{-2z}$ for $z \geq 0$. What is $\Pr\Big(Z \in [1, 2]\Big)$?

(Cumulative) distribution functions

- For a random variable X, its **(cumulative) distribution function** (cdf) $F(\cdot)$ is defined as

$$F_X(t) = \Pr(X \leq t)$$

  for all t in the sample space.
  - If X is continuous, then $F_X(t) = \int_{-\infty}^{t} f_X(x)dx$ and $f_X(x) = F'_X(x)$.
  - Let Y be the outcome of rolling a dice. What is $F_Y(y)$?
  - Let $Z \sim f_Z$ where $f_Z(z) = 2e^{-2z}$ for $z \geq 0$. What is $F_Z(z)$?

Expectations

- For a discrete random variable X whose sample space is S, its **expectation** (or **expected value**) $\mathbb{E}[X]$ is

$$\mathbb{E}[X] = \sum_{x \in S} x p_X(x).$$

  - What is the expectation of rolling a dice?

- For a continous random variable X whose sample space is S, its expectation $\mathbb{E}[X]$ is

$$\mathbb{E}[X] = \int_{x \in S} x f_X(x) dx.$$

  - Let $Y \sim f_Y$ where $f_Y(y) = \frac{1}{4}$ for $y \in [0, 4]$. What is $\mathbb{E}[Y]$?

IM 7011: Information Economics

Overview and Preliminaries
Lecture 1.3: Optimality Conditions

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Introduction

- Here we introduce **optimality conditions** for optimization problems.
- These conditions are critical for us to obtain **analytical solutions**.
- Only with analytical solutions we may deliver **business/economic implications**, or **insights**.

Road map

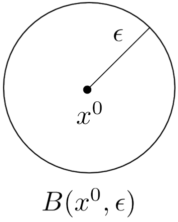
- **Optimality conditions for unconstrained problems**.
- Application: monopoly pricing.
- Application: the newsvendor problem.

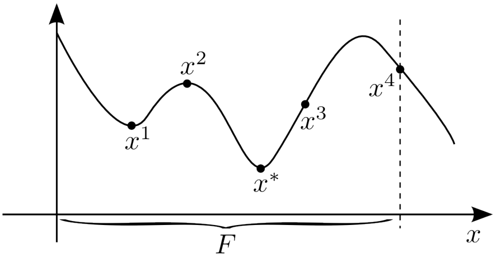
Global optima

- For a function $f(x)$ over a feasible region F:
  - A point x^* is a **global minimum** if $f(x^*) \le f(x)$ for all $x \in F$.
  - A point x' is a **local minimum** if for some $\epsilon > 0$ we have

$$f(x') \le f(x) \quad \forall x \in B(x', \epsilon) \cap F,$$

where $B(x^0, \epsilon) \equiv \{x | d(x, x^0) \le \epsilon\}$ and $d(x, y) \equiv \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.





- **Global maxima** and **local maxima** are defined accordingly.

First-order necessary condition

- Consider an **unconstrained** problem

$$\max_{x \in \mathbb{R}^n} f(x).$$

Proposition 1 (Unconstrained FONC)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable. For a point x^* to be a local maximum of f, we need:

- $f'(x^*) = 0$ *if* $n = 1$.
- $\nabla f(x^*) = 0$ *if* $n \geq 2$.

- For an n-dimensional differentiable function f, its **gradient** is

$$\nabla f \equiv \begin{bmatrix} \dfrac{\partial f}{\partial x_1} \\ \vdots \\ \dfrac{\partial f}{\partial x_n} \end{bmatrix}.$$

Examples

- Consider the problem

$$\max_{x \in \mathbb{R}} \ x^3 - 3x^2 + 4x + 2$$

The FONC yields

$$3(x^2 - 3x + 2) = 0.$$

Solving the equation gives us 1 and 2 as two candidates of local maxima.

  - It is easy to see that $x^* = 1$ is a local maxima but $\tilde{x} = 2$ is NOT.

- Consider the problem

$$\max_{x \in \mathbb{R}^2} \ f(x) = x_1^2 - x_1 x_2 + x_2^2 - 6x_2.$$

The FONC yields

$$\nabla f(x) = \begin{bmatrix} 2x_1 - x_2 \\ -x_1 + 2x_2 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving the linear system gives us $(2, 4)$ as the only candidate of local maxima.

  - Note that it may NOT be a local maximum!

Second-order necessary condition

- Let's proceed further.

Proposition 2 (Unconstrained SONC)

Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is twice-differentiable. For a point x^* to be a local maximum of f, we need:

- $f''(x^*) \leq 0$ *if* $n = 1$.
- $y^T \nabla^2 f(x^*) y \leq 0$ *for all* $y \in \mathbb{R}^n$ *if* $n \geq 2$.

- For an n-dimensional function $f(x_1, ..., x_n) : \mathbb{R}^n \to \mathbb{R}$ that is twice-differentiable, its **Hessian** is the $n \times n$ matrix

$$\nabla^2 f \equiv \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \, \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \, \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}.$$

Second-order necessary condition

- Regarding the Hessian:
  - (Calculus) If the second-order derivatives are all continuous (which will be true for almost all functions we will see in this course), the Hessian is symmetric.
  - (Linear Algebra) A symmetric matrix A is called **negative semidefinite** if $y^T Ay \leq 0$ for all $y \in \mathbb{R}^n$.
  - Therefore, if the second-order derivatives of f all exists and are continuous, the unconstrained SONC is simply requesting the Hessian to be negative semidefinite.
- In this course, we will not apply the SONC a lot.
- Here our point is that a local maximum requires NOT just

$$\frac{\partial^2 f}{\partial x_i^2} \leq 0 \quad \forall i = 1, ..., n.$$

We want more than candidates!

- The FONC and SONC produce candidates of local maxima/minima.
- What's next?
  - We need some ways to **ensure** local optimality.
  - We need to find a **global** optimal solution.
- While complicated methods exist for general functions, only simple conditions are required for concave/convex functions.
  - Because for a differentiable concave/convex function, the FONC is necessary AND **sufficient** (thus called FOC in this case).

- Now points satisfying the FONC are locally optimal.
- Our final step is to show that they are also **globally** optimal.

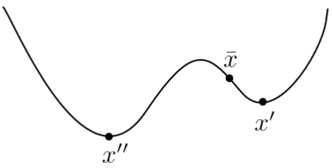
Local v.s. global optima

Proposition 3 (Global optimality of convex functions)

For a convex (concave) function f, a local minimum (maximum) is a global minimum (maximum).

Proof. Suppose a local min x' is not a global min and there exists x'' such that $f(x'') < f(x')$. Consider a small enough $\lambda > 0$ such that $\bar{x} = \lambda x'' + (1-\lambda)x'$ satisfies $f(\bar{x}) > f(x')$. Such $\bar{x}$ exists because x is a local min. Now, note that

$$\begin{aligned} f(\bar{x}) &= f\Big(\lambda x'' + (1-\lambda)x'\Big) \\ &> f(x') \\ &= \lambda f(x') + (1-\lambda) f(x') \\ &> \lambda f(x'') + (1-\lambda) f(x'), \end{aligned}$$



which violates the fact that $f(\cdot)$ is convex. Therefore, by contradiction, the local min x must be a global min. □

Remarks

- When you are asked to solve a problem:
  - First check whether the objective function is convex/concave. If so the problem may become much more easier.
- All the conditions for unconstrained problems apply to **interior** points of a feasible region.
- One common strategy for solving constrained problems proceeds in the following steps:
  - Ignore all the constraints.
  - Solve the unconstrained problem.
  - Verify that the unconstrained optimal solution satisfies all constraints.
- If the strategy fails, we seek for other ways.

Road map

- Optimality conditions for unconstrained problems.
- **Application: monopoly pricing**.
- Application: the newsvendor problem.

Monopoly pricing

- Suppose a monopolist sells a single product to consumers.
- Consumers are **heterogeneous** in their **willingness-to-pay**, or **valuation**, of this product.
  - One's valuation, x, lies on the interval $[0, b]$ uniformly.
  - He buys the product if and only if his valuation is above the price.
  - The total number of consumers is a.
  - Given a price p, in expectation how many consumers buy?
- The unit production cost is c.
- The seller chooses a unit price p to maximize her total profit.

Formulation

- The **endogenous** decision variable is p.
- The **exogenous** parameters are a, b, and c.
- The only constraint is $p \geq 0$.
- Let $\pi(p)$ be the profit under price p. What is $\pi(p)$?

- What is the complete problem formulation?

    - It is equivalent to **normalize** the population size a to 1.

Solving the problem

- Given that $\pi(p) = \frac{a}{b}(p - c)(b - p)$, let's show it is strictly concave:
  - $\pi'(p) =$
  - $\pi''(p) =$
- Great! Now let's ignore the constraint $p \geq 0$.
- Applying the FOC, what is the unconstrained optimal solution?
- Does p^* satisfy the ignored constraint? Is it globally optimal?

Comparative statics

- The optimal price $p^* = \frac{b+c}{2}$ tells us something:
  - p^* is increasing in the highest possible valuation b. Why?
  - p^* is increasing in the unit cost c. Why?
  - p^* has nothing to do with the total number of consumer a. Why?
- The optimal profit $\pi^* \equiv \pi(p^*) = \frac{a(b-c)^2}{4b}$.
  - π^* is decreasing in c. Why?
  - π^* is increasing in a. Why?
  - How is π^* affected by b? Guess!
  - Let's answer it:
- It is these **qualitative** business/economic implications that matters.
- Never forget to verify your solutions with your **intuition**.

Robustness

- We "**proved**" one thing: The seller will charge more and earn more when the unit cost goes up.
  - Does this depend on our model assumptions?
  - In particular, what if the distribution of consumer valuations is not uniform (i.e., the demand function is not linear)?
- Let's examine the **robustness** of this finding by **generalizing** our demand function.
  - Suppose the demand function $D(p)$ is twice-differentiable.
  - The profit function is
$$\pi(p) = (p - c)D(p).$$
  - To check concavity, note that $D''(p) = 2D'(p) + D''(p)(p - c)$ (verify it!).
  - As long as D is nonincreasing and concave, $\pi(p)$ is concave (why?).
  - Under this assumption, the FOC requires the optimal price p^* to satisfy
$$D(p^*) + D'(p^*)(p^* - c) = 0.$$

Robustness

- For the equation $D(p) + D'(p)(p - c) = 0$, how does c affect p?

- We have "proved" that our finding is not so restrictive: It is true as long as $D(\cdot)$ is nonincreasing and concave.
  - This generalization can go further.
- Avoid using unreasonable assumptions to prove "surprising" results!

Road map

- Optimality conditions for unconstrained problems.
- Application: monopoly pricing.
- **Application: the newsvendor problem**.

Newsvendor problem

- In some situations, sellers face **uncertain demands**.
- Consider a vendor of newspapers:
  - She does not know how many people will buy in a day.
  - She has only **one chance** to prepare newspapers (at, e.g., 4am).
  - Unsold newspapers become (almost) **valueless**.
- For **perishable** products, sellers solve **single-period** problems.
  - These are also called **one-shot** problems.
  - For durable goods, sellers solve multi-period problems.
- As a newsvendor, what should be in your mind?

Newsvendor model

- Let D be the uncertain demand.
- Let F and f be the distribution and density functions of D.
  - This time let's directly use a general model.
  - The only assumption here is that D is continuous and nonnegative.
  - The insights we obtain will also apply to discrete random demands.
- Let r be the unit retail price and c be the unit replenishment cost.
- We want to find an order quantity q that maximizes the expected total profit.

Formulation

- The sales quantity, given the demand D and order quantity q, is

$$\min\{D, q\},$$

  which is also random.
- With this, the expected profit is
- The only constraint is $q \geq 0$.
- What is the complete formulation?

Concavity of the cost function

- As usual, let's analyze the objective function first.
- The expected profit $\pi(q)$ is

$$\begin{aligned}\pi(q) &= r\mathbb{E}\Big[\min\{D,q\}\Big] - cq = r\int_0^\infty \min\{x,q\}f(x)dx - cq \\ &= r\Big\{\int_0^q xf(x)dx + \int_q^\infty qf(x)dx\Big\} - cq \\ &= r\Big\{\int_0^q xf(x)dx + q[1-F(q)]\Big\} - cq.\end{aligned}$$

- We then have

$$\pi'(q) = r\Big[qf(q) + 1 - F(q) - qf(q)\Big] - c = r\Big[1 - F(q)\Big] - c.$$

and

$$\pi''(q) = -rf(q) \leq 0.$$

Optimizing the order quantity

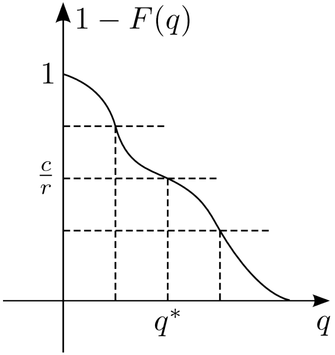
- So $\pi(q)$ is concave in q.
- Let q^* be the order quantity that satisfies the FOC, we have

$$\pi'(q^*) = r\Big[1 - F(q^*)\Big] - c = 0 \quad \Leftrightarrow \quad F(q^*) = 1 - \frac{c}{r}.$$

- As $0 < c < r$, we have $0 < 1 - \frac{c}{r} < 1$ and thus a reasonable q^* can be obtained (how?).
- As $D \geq 0$, q^* must be nonnegative. So q^* is optimal.

Trade-off between overage and underage

- Let's verify our solution with intuitions.
- The optimal probability of shortage is $1 - F(q^*) = \frac{c}{r}$.
  - When c goes up, creates a higher shortage probability by decreasing q^*.
  - When r goes up, creates a lower shortage probability by increasing q^*.
  - $\frac{c}{r}$ is called the **critical ratio**.
- Suppose the shape of F changes and $\mathbb{E}[D]$ goes up. Will q^* also go up?



Other components that may be modeled

- More components may be included in the model:
  - The unit salvage value for each unsold product.
  - The unit disposal fee for each unsold product.
  - The unit shortage cost for each unsatisfied customer.