

# Information Economics

## Introduction to Game Theory

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# Introduction

- ▶ Today we introduce **games** under **complete information**.
  - ▶ Complete information: All the information are publicly known.
  - ▶ They are **common knowledge**.
- ▶ We will introduce **static** and **dynamic** games.
  - ▶ Static games: All players act simultaneously (at the same time).
  - ▶ Dynamic games: Players act sequentially.
- ▶ We will illustrate the **inefficiency** caused by decentralization (lack of cooperation).
- ▶ We will show how to **solve** a game, i.e., to predict what players will do in **equilibrium**.

# Road map

- ▶ **Prisoners' dilemma.**
- ▶ Static games: Nash equilibrium.
- ▶ Cournot competition.
- ▶ Dynamic games: Backward induction.
- ▶ Pricing in a supply chain.

## Prisoners' dilemma: story

- ▶ A and B broke into a grocery store and stole some money. Before police officers caught them, they hid those money carefully without leaving any evidence. However, a monitor got their images when they broke the window.
- ▶ They were kept in two separated rooms. Each of them were offered two choices: **Denial or confession**.
  - ▶ If both of them deny the fact of stealing money, they will both get one month in prison.
  - ▶ If one of them confesses while the other one denies, the former will be set free while the latter will get nine months in prison.
  - ▶ If both confesses, they will both get six months in prison.
- ▶ They **cannot communicate** and they must make their choices **simultaneously**.
- ▶ All they want is to be in prison as short as possible.
- ▶ What will they do?

## Prisoners' dilemma: matrix representation

- ▶ We may use the following matrix to formulate this “game”:

		Player 2	
		Denial	Confession
Player 1	Denial	-1, -1	-9, 0
	Confession	0, -9	-6, -6

- ▶ There are two **players**, each has two possible **actions**.
- ▶ For each combination of actions, the two numbers are the **utilities** of the two players: the first for player 1 and the second for player 2.
- ▶ Prisoner 1 thinks:
  - ▶ “If he denies, I should confess.”
  - ▶ “If he confesses, I should still confess.”
  - ▶ “I see! I should confess anyway!”
- ▶ For prisoner 2, the situation is the same.
- ▶ The **solution** (outcome) of this game is that both will confess.

## Prisoners' dilemma: discussions

- ▶ In this game, confession is said to be a **dominant strategy**.
- ▶ This outcome can be “improved” if they can **cooperate**.
- ▶ **Lack of cooperation** can result in a **lose-lose** outcome.
  - ▶ Such a situation is **socially inefficient**.
- ▶ We will see more situations similar to the prisoners' dilemma.

## Solutions for a game

- ▶ Is it always possible to solve a game by finding dominant strategies?
- ▶ What are the solutions of the following games?

		Player 2	
		B	S
Player 1	B	2, 1	0, 0
	S	0, 0	1, 2

		Player 2	
		H	T
Player 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

- ▶ We need a new solution concept: Nash equilibrium!

## Road map

- ▶ Prisoners' dilemma.
- ▶ **Static games: Nash equilibrium.**
- ▶ Cournot competition.
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## Nash equilibrium: definition

- ▶ The most fundamental equilibrium concept is the **Nash equilibrium**:

### Definition 1

For an  $n$ -player game, let  $S_i$  be player  $i$ 's action space and  $u_i$  be player  $i$ 's utility function,  $i = 1, \dots, n$ . An action profile  $(s_1^*, \dots, s_n^*)$ ,  $s_i^* \in S_i$ , is a (pure-strategy) **Nash equilibrium** if

$$\begin{aligned} & u_i(s_1^*, \dots, s_{i-1}^*, s_i^*, s_{i+1}^*, \dots, s_n^*) \\ & \geq u_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*) \end{aligned}$$

for all  $s_i \in S_i$ ,  $i = 1, \dots, n$ .

- ▶ Alternatively,  $s_i^* \in \operatorname{argmax}_{s_i \in S_i} \left\{ u_i(s_1^*, \dots, s_{i-1}^*, s_i, s_{i+1}^*, \dots, s_n^*) \right\}$  for all  $i$ .
- ▶ In a Nash equilibrium, no one has an incentive to **unilaterally deviate**.
- ▶ The term “pure-strategy” will be explained later.

## Nash equilibrium: an example

- ▶ Consider the following game with no dominant strategy:

		Player 2		
		L	C	R
Player 1	T	0, 4	4, 0	5, 3
	M	4, 0	0, 4	5, 3
	B	3, 5	3, 5	6, 6

- ▶ What is a Nash equilibrium?
  - ▶ (T, L) is not: Player 1 will deviate to M or B.
  - ▶ (T, C) is not: Player 2 will deviate to L or R.
  - ▶ (B, R) is: No one will unilaterally deviate.
  - ▶ Any other Nash equilibrium?
- ▶ Why a Nash equilibrium is an “outcome”?
  - ▶ Imagine that they takes turns to make decisions until no one wants to move. What will be the outcome?

## Nash equilibrium: More examples

- ▶ Is there any Nash equilibrium of the prisoners' dilemma?

		Player 2	
		Denial	Confession
Player 1	Denial	-1, -1	-9, 0
	Confession	0, -9	-6, -6

- ▶ How about the following two games?

		Player 2	
		B	S
Player 1	B	2, 1	0, 0
	S	0, 0	1, 2

		Player 2	
		H	T
Player 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

## Existence of a Nash equilibrium

	H	T
H	1, -1	-1, 1
T	-1, 1	1, -1

- ▶ The last game does not have a “pure-strategy” Nash equilibrium.
- ▶ What if we allow **randomized** (mixed) strategy?
- ▶ In 1950, John Nash proved the following theorem regarding the **existence** of “mixed-strategy” Nash equilibrium:

### Proposition 1

*For a static game, if the number of players is finite and the action spaces are all finite, there exists at least one mixed-strategy Nash equilibrium.*

- ▶ This is a sufficient condition. Is it necessary?
- ▶ In most business applications of Game Theory, people focus only on pure-strategy Nash equilibria.

# Road map

- ▶ Prisoners' dilemma.
- ▶ Static games: Nash equilibrium.
- ▶ **Cournot competition.**
- ▶ Dynamic games: Backward induction.
- ▶ Pricing in a supply chain.

## Cournot Competition

- ▶ In 1838, Antoine Cournot introduced the following **quantity competition** between two retailers.
- ▶ Let  $q_i$  be the production quantity of firm  $i$ ,  $i = 1, 2$ .
- ▶ Let  $P(Q) = a - Q$  be the market-clearing price for an aggregate demand  $Q = q_1 + q_2$ .
- ▶ Unit production cost of both firms is  $c < a$ .
- ▶ Each firm wants to maximize its profit.
- ▶ Our questions are:
  - ▶ In this environment, what will these two firms do?
  - ▶ Is the outcome satisfactory?
  - ▶ What is the difference between duopoly and monopoly (i.e., decentralization and integration)?

## Cournot Competition

- ▶ Players: 1 and 2.
- ▶ Action spaces:  $S_i = [0, \infty)$  for  $i = 1, 2$ .
- ▶ Utility functions:

$$u_1(q_1, q_2) = q_1 \left[ a - (q_1 + q_2) - c \right] \text{ and}$$

$$u_2(q_1, q_2) = q_2 \left[ a - (q_1 + q_2) - c \right].$$

- ▶ As for an outcome, we look for a Nash equilibrium.
- ▶ If  $(q_1^*, q_2^*)$  is a Nash equilibrium, it must solve

$$q_1^* \in \operatorname{argmax}_{q_1 \in [0, \infty)} u_1(q_1, q_2^*) = \operatorname{argmax}_{q_1 \in [0, \infty)} q_1 \left[ a - (q_1 + q_2^*) - c \right] \text{ and}$$

$$q_2^* \in \operatorname{argmax}_{q_2 \in [0, \infty)} u_2(q_1^*, q_2) = \operatorname{argmax}_{q_2 \in [0, \infty)} q_2 \left[ a - (q_1^* + q_2) - c \right].$$

## Solving the Cournot competition

- ▶ For firm 1, we first see that the objective function is strictly concave:
  - ▶  $u_1'(q_1, q_2^*) = a - q_1 - q_2^* - c - q_1$ .
  - ▶  $u_1''(q_1, q_2^*) = -2 < 0$ .
- ▶ The FOC condition suggests  $q_1^* = \frac{1}{2}(a - q_2^* - c)$ .
  - ▶ If  $q_2^* < a - c$ ,  $q_1^*$  is optimal for firm 1.
- ▶ Similarly,  $q_2^* = \frac{1}{2}(a - q_1^* - c)$  is firm 2's optimal decision if  $q_1^* < a - c$ .
- ▶ So if  $(q_1^*, q_2^*)$  is a Nash equilibrium such that  $q_i^* < a - c$  for  $i = 1, 2$ , it must satisfy

$$q_1^* = \frac{1}{2}(a - q_2^* - c) \quad \text{and} \quad q_2^* = \frac{1}{2}(a - q_1^* - c).$$

- ▶ The unique solution to this system is  $q_1^* = q_2^* = \frac{a-c}{3}$ .
  - ▶ Does this solution make sense?
  - ▶ As  $\frac{a-c}{3} < a - c$ , this is indeed a Nash equilibrium. It is also unique.



## Distortion due to decentralization

- ▶ What is the “cost” of decentralization?
- ▶ Suppose the two firms' are **integrated** together to jointly choose the aggregate production quantity.
- ▶ They together solve

$$\max_{Q \in [0, \infty)} Q[a - Q - c],$$

whose optimal solution is  $Q^* = \frac{a-c}{2}$ .

- ▶ First observation:  $Q^* = \frac{a-c}{2} < \frac{2(a-c)}{3} = q_1^* + q_2^*$ .
- ▶ Why does a firm intend to **increase** its production quantity under decentralization?

## Inefficiency due to decentralization

- ▶ May these firms improve their profitability with integration?
- ▶ Under decentralization, firm  $i$  earns

$$\pi_i^D = \frac{(a-c)}{3} \left[ a - \frac{2(a-c)}{3} - c \right] = \left( \frac{a-c}{3} \right) \left( \frac{a-c}{3} \right) = \frac{(a-c)^2}{9}.$$

- ▶ Under integration, the two firms earn

$$\pi^C = \frac{(a-c)}{2} \left[ a - \frac{a-c}{2} - c \right] = \left( \frac{a-c}{2} \right) \left( \frac{a-c}{2} \right) = \frac{(a-c)^2}{4}.$$

- ▶  $\pi^C > \pi_1^D + \pi_2^D$ : The integrated system is more **efficient**.
- ▶ Through appropriate profit splitting, both firm earns more.
  - ▶ Integration can result in a **win-win** solution for firms!
- ▶ However, under monopoly the aggregate quantity is lower and the price is higher. Consumers **benefits** from **firms' competition**.

## The two firms' prisoners' dilemma

- ▶ Now we know the two firms should together produce  $Q = \frac{a-c}{2}$ .
- ▶ What if we suggest them to produce  $q'_1 = q'_2 = \frac{a-c}{4}$ ?
- ▶ This maximizes the total profit but is **not** a Nash equilibrium:
  - ▶ If he chooses  $q' = \frac{a-c}{4}$ , I will move to

$$q'' = \frac{1}{2}(a - q' - c) = \frac{3(a-c)}{8}.$$

- ▶ So both firms will have incentives to unilaterally deviate.
- ▶ These two firms are engaged in a prisoners' dilemma!

# Road map

- ▶ Prisoners' dilemma.
- ▶ Static games: Nash equilibrium.
- ▶ Cournot competition.
- ▶ **Dynamic games: Backward induction.**
- ▶ Pricing in a supply chain.

## Dynamic games

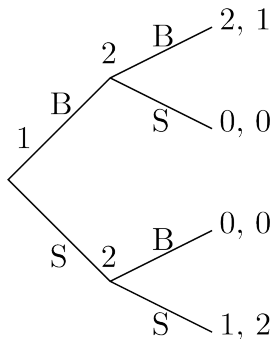
- ▶ Recall the game “BoS”:

		Player 2	
		B	S
		-----	
Player 1	B	2, 1	0, 0
	-----		
	S	0, 0	1, 2

- ▶ What if the two players make decisions **sequentially** rather than simultaneously?
  - ▶ What will they do in equilibrium?
  - ▶ How do their payoffs change?
  - ▶ Is it better to be the **leader** or the **follower**?

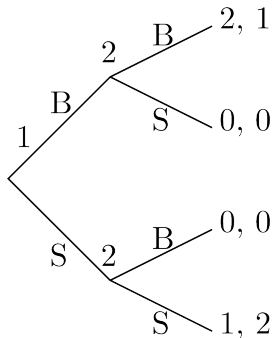
## Game tree for dynamic games

- ▶ Suppose player 1 **moves** first.
- ▶ Instead of a game matrix, the game can now be described by a **game tree**.
  - ▶ At each internal node, the label shows who is making a decision.
  - ▶ At each link, the label shows an action.
  - ▶ At each leaf, the numbers show the payoffs.
- ▶ The game is played from the root to leaves.



## Optimal strategies

- ▶ How should player 1 move?
- ▶ She must **predict** how player 2 will response:
  - ▶ If B has been chosen, choose B.
  - ▶ If S has been chosen, choose S.
- ▶ This is player 2's **best response**.
- ▶ Player 1 can now make her decision:
  - ▶ If I choose B, I will end up with 2.
  - ▶ If I choose S, I will end up with 1.
- ▶ So player 1 will choose B.
- ▶ An **equilibrium outcome** is a “path” goes from the root to a leaf.
  - ▶ In equilibrium, they play (B, B).



## Sequential moves vs. simultaneous moves

- ▶ In the static version, there are two pure-strategy Nash equilibria:
  - ▶ (B, B) and (S, S).
- ▶ When the game is played dynamically with player 1 moves first, there is only one **equilibrium outcome**:
  - ▶ (B, B).
- ▶ Their **equilibrium behaviors** change. Is it always the case?
- ▶ Being the leader is beneficial. Is it always the case?

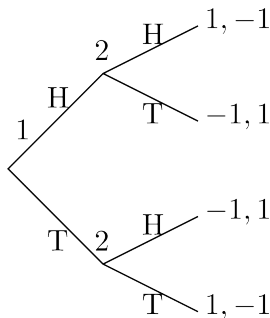


## Dynamic matching pennies

- ▶ Suppose the game “matching pennies” is played dynamically:

		Player 2	
		H	T
Player 1	H	1, -1	-1, 1
	T	-1, 1	1, -1

- ▶ What is the equilibrium outcome?
- ▶ There are multiple possible outcomes.
- ▶ Being the leader **hurts** player 1.



## Backward induction

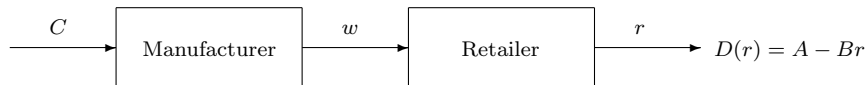
- ▶ In the previous two examples, there are a leader and a follower.
- ▶ Before the leader can make her decision, she must anticipate what the follower will do.
- ▶ When there are multiple **stages** in a dynamic game, we generally analyze those decision problems **from the last stage**.
  - ▶ The second last stage problem can be solved by having the last stage behavior in mind.
  - ▶ Then the third last stage, the fourth last stage, ...
- ▶ In general, we move **backwards** until the first stage problem is solved.
- ▶ This solution concept is called **backward induction**.

## Road map

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- ▶ **Pricing in a supply chain.**

## Pricing in a supply chain

- ▶ There is a manufacturer and a retailer in a supply chain.



- ▶ The manufacturer supplies to the retailer, who then sells to consumers.
- ▶ The manufacturer sets the **wholesale price**  $w$  and then the retailer sets the **retail price**  $r$ .
- ▶ The demand is  $D(r) = A - Br$ , where  $A$  and  $B$  are known constants.
- ▶ The unit production cost is  $C$ , a known constant.
- ▶ Each of them wants to maximize her or his profit.

## Pricing in a supply chain (illustrative)

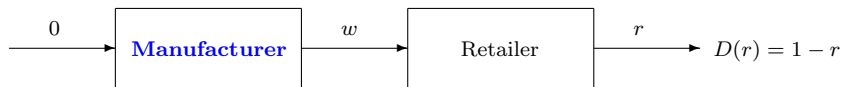


- ▶ Let's assume  $A = B = 1$  and  $C = 0$  for a while.
- ▶ Let's apply backward induction to **solve** this game.
- ▶ For the retailer, the wholesale price is **given**. He solves

$$\max_{r \geq 0} (r - w)(1 - r).$$

- ▶ The optimal solution (best response) is  $r^*(w) \equiv \frac{w+1}{2}$ .

## Pricing in a supply chain (illustrative)



- ▶ The manufacturer **predicts** the retailer's decision:
  - ▶ Given her offer  $w$ , the retail price will be  $r^*(w) \equiv \frac{w+1}{2}$ .
  - ▶ More importantly, the **order quantity** (which is the demand) will be

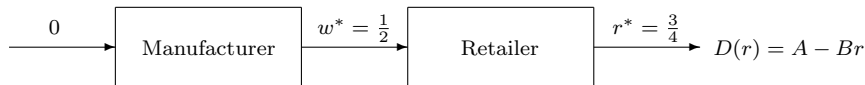
$$1 - r^*(w) = 1 - \frac{w+1}{2} = \frac{1-w}{2}.$$

- ▶ The manufacturer's solves

$$\max_{w \geq 0} w \left( \frac{1-w}{2} \right).$$

- ▶ The optimal solution is  $w^* = \frac{1}{2}$ .

## Pricing in a supply chain (illustrative)



- ▶ As the manufacturer offers  $w^* = \frac{1}{2}$ , the resulting retail price is

$$r^* \equiv r^*(w^*) = \frac{w^* + 1}{2} = \frac{3}{4} > \frac{1}{2} = w^*.$$

- ▶ A common practice called **markup**.
- ▶ The **sales volume** is  $D(r^*) = 1 - r^* = \frac{1}{4}$ .
- ▶ The retailer earns  $(r^* - w^*)D(r^*) = (\frac{1}{4})(\frac{1}{4}) = \frac{1}{16}$ .
- ▶ The manufacturer earns  $w^*D(r^*) = (\frac{1}{2})(\frac{1}{4}) = \frac{1}{8}$ .
- ▶ In total, they earn  $\frac{1}{16} + \frac{1}{8} = \frac{3}{16}$ .

## Pricing in a supply chain (general)

- ▶ For the retailer, the wholesale price is given. He solves

$$\max_{r \geq 0} (r - w)(A - Br)$$

- ▶ The optimal solution is  $r^*(w) \equiv \frac{Bw+A}{2B}$ .
- ▶ The manufacturer predicts the retailer's decision:
  - ▶ Given her offer  $w$ , the retail price will be  $r^*(w) \equiv \frac{Bw+A}{2B}$ .
  - ▶ More importantly, the order quantity (which is the demand) will be  $A - Br^*(w) = A - \frac{Bw+A}{2} = \frac{A-Bw}{2}$ .
- ▶ The manufacturer's problem:

$$\max_{w \geq 0} (w - C) \left( \frac{A - Bw}{2} \right)$$

- ▶ The optimal solution is  $w^* = \frac{BC+A}{2B}$ .



## Pricing in a supply chain (general)

- ▶ As the manufacturer offers  $w^* = \frac{BC+A}{2B}$ , the resulting retail price is  $r^* \equiv r^*(w^*) = \frac{Bw^*+A}{2B} = \frac{BC+3A}{4B}$ .
- ▶ The sales volume is  $D(r^*) = A - Br^* = \frac{A-BC}{4}$ .
- ▶ The retailer earns  $(r^* - w^*)D(r^*) = \left(\frac{A-BC}{4B}\right)\left(\frac{A-BC}{4}\right) = \frac{(A-BC)^2}{16B}$ .
- ▶ The manufacturer earns  $(w^* - C)D(r^*) = \left(\frac{A-BC}{2B}\right)\left(\frac{A-BC}{4}\right) = \frac{(A-BC)^2}{8B}$ .
- ▶ In total, they earn  $\frac{(A-BC)^2}{16B} + \frac{(A-BC)^2}{8B} = \frac{3(A-BC)^2}{16B}$ .

## Pricing in a cooperative supply chain

- ▶ Suppose the two firms are **cooperative**.
- ▶ They decide the wholesale and retail prices together.
- ▶ Is there a way to allow both players to be **better off**?
- ▶ Consider the following proposal:
  - ▶ Let's set  $w^{\text{FB}} = C = 0$  and  $r^{\text{FB}} = \frac{1}{2}$  (FB: **first best**).
  - ▶ The sales volume is

$$D(r^{\text{FB}}) = 1 - \frac{1}{2} = \frac{1}{2}.$$

- ▶ The total profit is
$$r^{\text{FB}} D(r^{\text{FB}}) = \frac{1}{4}.$$
- ▶ This is **larger** than  $\frac{3}{16}$ , the total profit generated under decentralization.
- ▶ How to split the pie to get a **win-win** situation?