Signaling Games with a Continuous Principal's Action Space

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1 Model

Consider the following signaling problem with a continuous decision space. A manufacturer sells a product of hidden reliability r to a consumer. We have $r \in \{r_L, r_H\}$, and the consumer's prior belief on r is $\Pr(r = r_L) = \beta = 1 - \Pr(r = r_H)$. The manufacturer chooses a price $t \in \mathbb{R}$ and a warranty protection probability $w \in [0, 1]$. By selling the product, the type-i manufacturer's expected utility is

$$u_i^M(t, w) = t - (1 - r_i)wk,$$

where k is the cost of fixing a broken product. By buying the product with r as the expected reliability, the consumer's expected utility is

$$u^C = r\theta + (1 - r)\eta w - t,$$

where θ is the utility of using a functional product and η is the utility of using a fixed product. We assume that

$$k > \eta \quad \text{and} \quad \theta > \eta.$$

2 Analysis

2.1 First best

Assume that r is public, let's find the type-i manufacturer's first-best offer (t_i^{FB}, w_i^{FB}) with reliability r_i. The manufacturer's problem is

$$\begin{aligned} \max_{t \in \mathbb{R}, w \in [0,1]} \quad & t - (1 - r_i)wk \\ \text{s.t.} \quad & r\theta + (1 - r_i)\eta w - t \geq 0, \end{aligned}$$

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which reduces to looking for $w \in [0,1]$ to maximize $r\theta + (1-r_i)(\eta - k)w$. As $\eta < k$, we have $w^{FB} = 0$ and thus $t^{FB} = r_i\theta$. Note that both types of manufacturers have no incentive to offer a warranty, and the high-type manufacturer earns more.

2.2 Second best

Assume that r is private, let's find the high-type manufacturer's offer (t_H^*, w_H^*) in a separating equilibrium. Suppose that the low-type manufacturer chooses its first-best offer $(t_i^*, w_i^*) = (r_L\theta, 0)$. The high-type manufacturer's problem is

$$\begin{aligned} \max_{t_H \in \mathbb{R}, w_H \in [0,1]} \quad & t_H - (1-r_H)w_H k \\ \text{s.t.} \quad & r_H\theta + (1-r_H)\eta w_H - t_H \geq 0 && \text{(IR)} \\ & t_L^* - (1-r_L)w_L^* k \geq t_H - (1-r_L)w_H k && \text{(IC-L)} \\ & t_H - (1-r_H)w_H k \geq t_L^* - (1-r_H)w_L^* k. && \text{(IC-H)} \end{aligned}$$

By replacing t_i^* and w_i^* by $r_L\theta$ and 0, the problem reduces to

$$\begin{aligned} \max_{t_H \in \mathbb{R}, w_H \in [0,1]} \quad & t_H - (1-r_H)w_H k \\ \text{s.t.} \quad & r_H\theta + (1-r_H)\eta w_H - t_H \geq 0 && \text{(IR)} \\ & r_L\theta \geq t_H - (1-r_L)w_H k && \text{(IC-L)} \\ & t_H - (1-r_H)w_H k \geq r_L\theta. && \text{(IC-H)} \end{aligned}$$

Let's ignore (IC-H) for a while. Suppose that (IC-L) is not binding, then (IR) is binding, and the problem reduces to

$$\max_{w_H \in [0,1]} \; r_H\theta + (1-r_H)(\eta - k)w_H,$$

and the optimal solution is $w_H = 0$ and $t_H = r_H\theta$. This violates (IC-L), so we know (IC-L) must be binding. Therefore, the problem reduces to

$$\begin{aligned} \max_{w_H \in [0,1]} \quad & r_L\theta + (r_H - r_L)w_H k \\ \text{s.t.} \quad & r_H\theta + (1-r_H)\eta w_H - \Big[r_L\theta + (1-r_L)w_H k\Big] \geq 0, && \text{(IR)} \end{aligned}$$

where the (IR) constraint is equivalent to

$$(r_H - r_L)\theta + \Big[(1-r_H)\eta - (1-r_L)k\Big]w_H \geq 0.$$

Note that as $r_H > r_L$ and $k > \eta$, we have $(1-r_H)\eta - (1-r_L)k < 0$, and thus w_H is bounded above. Moreover, the objective function is clearly maximized when $w_H = 1$. Therefore, we have

$$w_H^* = \min\left\{1, \frac{(r_H - r_L)\theta}{(1-r_L)k - (1-r_H)\eta}\right\} \qquad t_H^* = r_L\theta + (1-r_L)w_H^* k.$$

It is straightforward to verify that (IC-H) is satisfied.