Linear Algebra and its Applications, Spring 2013
Final Exam

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Note. In total there are 110 points, but you may get at most 100. Do not return the problem sheet.

1. (10 points; 5 points each) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 5 & 1 & 0 & 4 \\ 1 & 2 & 4 & 0 & -3 & 4 \\ 3 & 6 & 14 & 4 & -3 & 13 \\ 2 & 4 & 10 & 0 & 0 & 11 \end{bmatrix}$.

   (a) Do the LU decomposition for A. Perform row exchanges and write down the P matrix when necessary. Do the LU decomposition, not the LDU decomposition.

   (b) Project $d = (3, 2, 4, 5)$ onto the column space of A.

2. (15 points) Consider the following two problems regarding determinants.

   (a) (5 points) Suppose an $n \times n$ matrix A can be decomposed into $A = Q\Lambda Q^T$, where Q is the orthonormal matrix of eigenvectors of A and the diagonal elements of Λ are A's eigenvalues. By using the equation $A = Q\Lambda Q^T$, show that the determinant of A is the product of A's eigenvalues.

   (b) (10 points) Find the determinants of $B \in \mathbb{R}^{4\times 4}$ and $C \in \mathbb{R}^{n\times n}$, where B and C satisfy

$$B = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 1 & 0 & 3 & 4 \\ 1 & 2 & 0 & 4 \\ 1 & 2 & 3 & 0 \end{bmatrix} \quad \text{and} \quad C_{ij} = \begin{cases} 0 & \text{if } i = j \\ j & \text{if } i \neq j \end{cases}.$$

3. (15 points) Consider matrix $A = \begin{bmatrix} 0.2 & 0.5 & 0.2 \\ 0.3 & 0.5 & 0.3 \\ 0.5 & 0 & 0.5 \end{bmatrix}$. Let $u_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $u_k = Au_{k-1}$, $k = 1, 2, ...$

   (a) (2 points) Find u_1.

   (b) (2 points) Find u_2.

   (c) (2 points) Prove that the process $\{u_k\}_{k=1,2,...}$ is neutrally stable WITHOUT finding all the eigenvalues of A.

   (d) (5 points) Find u_∞, which satisfies $u_\infty = Au_\infty$.

   (e) (2 points) Find the first principal component of A for a given data point $x \in \mathbb{R}^3$.

   (f) (2 points) Find the first principal component of A^2 for a given data point $x \in \mathbb{R}^3$.

4. (15 points; 3 points each) For each of the following questions regarding an $n \times n$ real matrix A, answer either "True" or "False". Do not provide any explanation.

   (a) If all the eigenvalues of A are distinct, then A can be decomposed into $A = Q\Lambda Q^T$, where Λ is diagonal and Q is orthornormal.

   (b) A is symmetric if and only if A's eigenvalues are all real.

   (c) If AA^T is nonsingular, then A^TA is nonsingular.

   (d) If A is unitary, there can be n independent eigenvectors of A.

   (e) If A's eigenvalues are all positive, A can be decomposed into $A = R^TR$, where R has n pivots.

5. (10 points) Find the singular value decomposition for $A = \begin{bmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix}$.

6. (20 points) Let A be an $n \times n$ positive semidefinite matrix. Suppose A has distinct eigenvalues λ_1, λ_2, ..., λ_n.

   (a) (5 points) Prove or disprove that A's eigenvectors are all orthogonal.

   (b) (2 points) Find a lower bound of A's eigenvalues. Briefly explain why. You CANNOT use $\min_{i=1,\dots,n}\{\lambda_i\}$ as your answer.

   (c) (4 points) Given a constant k, prove or disprove that the eigenvalues of the matrix $A - kI$ are $\lambda_1 - k$, $\lambda_2 - k$, ..., $\lambda_n - k$.

   (d) (4 points) Consider a nonlinear program that looks for a vector $x \in \mathbb{R}^n$ to minimize

$$f(x) = x^T Ax - kx^T x,$$

   where k is a given constant in $\mathbb{R}$. Prove or disprove that x is a stationary point of $f(x)$ if and only if x is an eigenvector of A.

   (e) (5 points) Suppose now we look for $(x, k) \in \mathbb{R}^n \times \mathbb{R}$ to minimize $f(x, k) = x^T Ax - kx^T x$. In other words, now k is also a decision variable. Among all the stationary points, who are local minima? Prove your claim.

7. (10 points) Suppose there is a standard form linear program that minimizes $c^T x$ subject to $Ax = b$ and $x \geq 0$. Suppose $x \in \mathbb{R}^n$ and A is an $m \times n$ matrix where $m \leq n$. Moreover, $b \geq 0$ is an $m \times 1$ column vector. Assume that the rows of A are linearly independent.

   (a) (2 points) Suppose $m = 2$ and $n = 5$, find an upper bound of the number of basic solutions. Prove your claim.

   (b) (2 points) Suppose $m = 3$ and $n = 6$, find an upper bound of the number of basic feasible solutions. Prove your claim.

   (c) (3 points) Suppose $A = \begin{bmatrix} 2 & 0 & 1 & 1 & 0 \\ 0 & 4 & 1 & 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$. Find the number of distinct bases. You do not need to list all of them.

   Note. A basis is a collection of m basic variables that corresponds to a basic solution. Two different bases may correspond to *the same* basic solution. In this case, however, they are still treated as distinct bases.

   (d) (3 points) For general m and n, is the number of basic feasible solutions always smaller than the number of basic solution? Prove it or find a counterexample.

8. (15 points) Consider a firm selling n products whose demand quantities are determined by the prices. For product i, $i = 1, ..., n$, the demand quantity is

$$q_i = a_i - rp_i + \sum_{j \neq i} p_j,$$

where $a_i > 0$ and $r > 0$ are given constants. Please note that the demand of one product decreases when its price increases but increases when *other* products become more expensive. For simplicity, we assume that the production cost is 0 for all products, so the firm's *negative* profit function is

$$\pi(p) = -\sum_{i=1}^{n} p_i \left(a_i - rp_i + \sum_{j \neq i} p_j \right).$$

The nonlinear program to be solved by the firm is to find $p \in \mathbb{R}^n$ to minimize $\pi(p)$ subject to $p \geq 0$.

   (a) (2 points) Suppose $n = 2$, find the Hessian matrix of $\pi(p)$.

   (b) (2 points) Suppose $n = 2$, find an interval of r such that $\pi(p)$ is strictly convex if and only if r is within that interval. Prove your claim.

(c) (3 points) Suppose $n = 3$, find the Hessian matrix of $\pi(p)$.

(d) (3 points) Suppose $n = 3$, find an interval of r such that $\pi(p)$ is strictly convex if and only if r is within that interval. Prove your claim.

(e) (5 points) For a general n, find an interval of r such that $\pi(p)$ is strictly convex if and only if r is within that interval. Prove your claim.