Linear Algebra and its Applications, Spring 2013
Suggested Solution for Midterm Exam

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1. (a) We have

$$A = \begin{bmatrix} 1 & 2 & 5 & 1 & 0 & 4 \\ 1 & 2 & 4 & 0 & -3 & 4 \\ 3 & 6 & 14 & 4 & -3 & 13 \\ 2 & 4 & 10 & 0 & 0 & 11 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 \\ 2 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 & 1 & 0 & 4 \\ 0 & 0 & -1 & -1 & -3 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix} = LU.$$

(b) According to Part (c), a nullspace solution can be expressed as

$$k(-2,1,0,0,0,0) + r(15,0,-3,0,1,0), \quad k, r \in \mathbb{R}.$$

It then suffices to find a particular solution. To do so, we may solve

$$\begin{bmatrix} 1 & 5 & 1 & 4 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \\ -2 \\ -4 \end{bmatrix},$$

where the matrix is obtained by collecting pivot columns in U and the RHS vector is obtained by performing the same elementary row operations reducing A to U to $b = (1,0,0,0)$. The unique solution that solves the above equation is $x = (-2, \frac{3}{2}, -\frac{1}{2}, -1)$, which can be translated into a particular solution $(-2, 0, \frac{3}{2}, -\frac{1}{2}, 0, -1)$. All the solutions can then be expressed as

$$\left(-2, 0, \frac{3}{2}, -\frac{1}{2}, 0, -1\right) + k(-2,1,0,0,0,0) + r(15,0,-3,0,1,0), \quad k, r \in \mathbb{R}.$$

(c) The basis of the column space is the four columns in A that corresponds to the four pivot columns in U. As the rank of the column space is 4, the rank of the left nullspace is 0. The basis of the row space is the four pivot rows in U. To find the nullspace, we reduce A to

$$R = \begin{bmatrix} 1 & 2 & 0 & 0 & -15 & 0 \\ 0 & 0 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The basis of the four fundamental subspaces are listed below:

$$\text{basis}\Big(\mathcal{R}(A)\Big) = \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 14 \\ 10 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 13 \\ 11 \end{bmatrix} \right\}, \text{basis}\Big(\mathcal{N}(A^T)\Big) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

$$\text{basis}\Big(\mathcal{R}(A^T)\Big) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \\ 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}, \text{basis}\Big(\mathcal{N}(A)\Big) = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -15 \\ 0 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(d) Because the column space is the entire $\mathbb{R}^4$, the projection of any vector in $\mathbb{R}^4$ onto the column space is the same vector. Therefore, the projection of d is $(3,2,4,5)$.

2. (a) As the determinant of A_1 can be found by plugging $x = 1$ into the determinant of A_2, let's calculate the determinant of A_2. We have

$$\det A_2 = 2\begin{vmatrix} 4 & 2 & 1 \\ 0 & x & 5 \\ 2 & 3 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 4 & 1 \\ 0 & 0 & 5 \\ 0 & 2 & 0 \end{vmatrix} = 2x\begin{vmatrix} 4 & 1 \\ 2 & 0 \end{vmatrix} - 10\begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} = -4x - 80.$$

Therefore, $\det A_1 = -80$.

(b) Because A_2 is square, the dimension of the left nullspace of A_2 is greater than 0 if and only if $\det A_2 = 0$. This requires $x = -20$.

3. (a) A right inverse is

$$\begin{bmatrix} 0.4 & -0.1 \\ 0 & 0 \\ -0.2 & 0.3 \\ 0 & 0 \end{bmatrix}$$

(b) There is no left inverse. To see this, note that if B is a left inverse, it must satisfy

$$B\begin{bmatrix} 3 & 0 & 1 & 0 \\ 2 & 0 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

However, as the second column of BA must be 0, the above equation cannot be satisfied.

4. (a) True. Suppose b_1 and b_2 are in the column space, there exist x_1 and x_2 such that $Ax_1 = b_1$ and $Ax_2 = b_2$. Given any $\lambda \in [0, 1]$, we have

$$\lambda b_1 + (1 - \lambda)b_2 = \lambda Ax_1 + (1 - \lambda)Ax_2 = A(\lambda x_1 + (1 - \lambda)x_2).$$

This implies that $\lambda b_1 + (1 - \lambda)b_2$ is in the column space of A.

(b) True. Suppose $A = A^T$ and $B = B^T$, then for any $\lambda \in [0, 1]$, we have

$$\lambda A + (1 - \lambda)B = \lambda A^T + (1 - \lambda)B^T = (\lambda A + (1 - \lambda)B)^T.$$

This implies that $\lambda A + (1 - \lambda)B$ is symmetric.

(c) (4 points) The set of 3×3 singular matrices is a convex set.
False. To see this, consider

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \text{ and } \frac{1}{2}A + \frac{1}{2}B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is clear that both A and B are singular but $\frac{1}{2}A + \frac{1}{2}B$ is nonsingular.

5. (a) The column space of A is contained in the column of space. To see this, consider a vector $b \in \mathbb{R}^m$ in the column space of $A \in \mathbb{R}^{m\times n}$. In this case, there exists $x \in \mathbb{R}^n$ such that $Ax = b$. Now consider $y = (x, 0) \in \mathbb{R}^{n+1}$. It is clear that $Cy = Ax + 0 = Ax = b$, and thus b is in the column space of C. The reverse is not true and a counterexample can be easily obtained.

(b) Neither one contains the other. To see this, note that the nullspace of A is a subspace of $\mathbb{R}^n$ while the nullspace of C is a subspace of $\mathbb{R}^{n+1}$. It is impossible for a subspace of $\mathbb{R}^k$ to contain a subspace of $\mathbb{R}^p$ if $k \neq p$.

(c) The left nullspace of C is contained in the left nullspace of A. To see this, we only need to combine the fact that the column space and left nullspace of any matrix are orthogonal complements and the result in Part (a). Note that the statement can certainly be proved directly.

6. (a) ab^T is a m by n matrix.
   (b) $\text{rank}(ab^T) = 1$.
   (c) The answer is the column space of ab^T, which is $\{x \in \mathbb{R}^m | x = ka, k \in \mathbb{R}\}$.
   (d) The matrix A must be $n \times m$. The transformation is in general not invertible. For example, when $b = 0$ but $a \neq 0$, this is an n-to-1 mapping, which is not invertible.

7. (a) In general, we should find the projection of c onto a and b, subtract the two projections from c, and then normalize the resulting vector. However, because a and b are two independent vectors in $\mathbb{R}^3$, there is only one direction for q to be orthogonal to both a and b. All we need to do is to solve

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 1 & 1 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

   find any solution, and normalize the solution to be of length 1. A solution to the above equation is $(1, 5, -4)$, and normalization results in $q = (\frac{1}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-4}{\sqrt{42}})$.
   (b) Because each row of A is orthogonal to each other, we know in AA^T all the off-diagonal entries will be 0. Moreover, the diagonal entries will be $a^Ta = 14$, $b^Tb = 3$, and $q^Tq = 1$. Therefore, $\det AA^T = 14 \times 3 \times 1 = 42$.[1]

8. Suppose x is in the nullspace of A, then $Ax = 0$, which implies $A^TAx = 0$ and thus x is in the nullspace of A^TA. Suppose x is in the nullspace of A^TA, then $A^TAx = 0$, which implies $x^TA^TAx = 0$, $||Ax|| = 0$, or $Ax = 0$. This means that x is in the nullspace of A.

9. (10 points) Consider the following transformations. For each of them, prove or disprove that it is a linear transformation. If it is, find a matrix A that performs the linear transformation.

   (a) The transformation is linear and the matrix is

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

   (b) (3 points) The transformation that maps a square matrix to its transpose.
   Let X and Y be two square matrices of the same dimension, a and b be two scalars, and $T(\cdot)$ be the transformation, then

$$T(aX + bY) = (aX + bY)^T = aX^T + bY^T = aT(X) + b(Y),$$

   which implies that the transformation is linear. To find the matrix that does the transformation, consider 2×2 square matrices as an example. In this case, if we want to express the transformation by a matrix, we need to first express each 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

   as a vector $(a, b, c, d) \in \mathbb{R}^4$.[2] The matrix A that does the transformation is thus 4×4 and should satisfy

$$A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix} \quad \Leftrightarrow \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

   The same methodology applies to $n \times n$ matrices for any $n \in \mathbb{N}$.
   (c) (4 points) The transformation that maps a matrix $A \in \mathbb{R}^{2\times 2}$ to $(d_1, d_2) \in \mathbb{R}^2$, where d_1 and d_2 are the pivots of A. As an example, if we call the transformation T, then
   This transformation is not linear. To see this, note that

$$T\left(\begin{bmatrix} 3 & 1 \\ 6 & 1 \end{bmatrix}\right) + T\left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \neq T\left(\begin{bmatrix} 2 & 1 \\ 6 & 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

[1] Please note that solving Part (b) does not require solving Part (a)!
[2] Different ways to arrange the numbers are also allowed. The answer depends on how these numbers are arranged.

10. (10 points; 5 points each) Solve the following two problems:

(a) The inverse is

$$\begin{bmatrix} 1 & -2 & 6 & 19 \\ 0 & 1 & -4 & -15 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) The solution is

$$(A^T A)^{-1} A b = \begin{bmatrix} \frac{13}{6} \\ \frac{5}{6} \\ \frac{3}{2} \end{bmatrix}.$$