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Suggested Solution for Final Exam

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. We may solve this three-variable linear program with the simplex method directly. However, we may simplify our calculation by first observing that in an optimal solution, if there is any, we have $x_3 = 3$. The problem can then be reduced to

$$\begin{array}{rrcrcl} \max & 2x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \geq & 3 \\ & x_1 & - & 2x_2 & \geq & 2 \\ & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2.} \end{array}$$

To solve this linear program, we first construct the Phase-I linear program

$$\begin{array}{rrcrcrcrcrcrcl} \max & & & & & & - & x_4 & & & - & x_6 & & \\ \text{s.t.} & x_1 & + & x_2 & - & x_3 & + & x_4 & & & & & = & 3 \\ & x_1 & - & 2x_2 & & & & & - & x_5 & + & x_6 & = & 2 \\ & \multicolumn{12}{l}{x_i \geq 0 \quad \forall i = 1, ..., 6.} \end{array}$$

We fix the objective row, obtain the initial tableau, do two iterations, and solve the Phase-I linear program as follows.[1]

$$\begin{array}{rrrrrr|c} -2 & 1 & 1 & 0 & 1 & 0 & -5 \\ \hline 1 & 1 & -1 & 1 & 0 & 0 & x_4 = 3 \\ \boxed{1} & -2 & 0 & 0 & -1 & 1 & x_6 = 2 \end{array} \rightarrow \begin{array}{rrrrr|c} 0 & -3 & 1 & 0 & -1 & -1 \\ \hline 0 & 3 & -1 & 1 & \boxed{1} & x_4 = 1 \\ 1 & -2 & 0 & 0 & -1 & x_1 = 2 \end{array} \rightarrow \begin{array}{rrrr|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 3 & -1 & 1 & x_5 = 1 \\ 1 & 1 & -1 & 0 & x_1 = 3 \end{array}$$

As we have found an initial basic feasible solution, we recover the objective function:

$$\begin{array}{rrrr|c} -2 & -1 & 0 & 0 & 0 \\ \hline 0 & 3 & -1 & 1 & x_5 = 1 \\ 1 & 1 & -1 & 0 & x_1 = 3 \end{array}$$

After we fix the objective row to obtain a valid tableau

$$\begin{array}{rrrr|c} 0 & 1 & -2 & 0 & 6 \\ \hline 0 & 3 & -1 & 1 & x_5 = 1 \\ 1 & 1 & -1 & 0 & x_1 = 3 \end{array}$$

we find that the linear program is unbounded (due to column 3). Therefore, the original linear program is unbounded.

2. (a) For this program to be a convex program, we need $f(y)$ be a convex function and $g(y)$ be a convex function.

   (b) The Lagrangian relaxation is

$$\begin{array}{rl} \min & f(y) + \lambda g(y) \\ \text{s.t.} & 0 \leq y_j \leq 1 \quad \forall j = 1, ..., 4. \end{array}$$

   for some $\lambda \geq 0$.

[1] Note that we may also choose to enter x_2 in the second iteration. The final conclusion will not change. Here we enter x_5 because the calculation will be easier.

(c) As the original program is quite similar to the continuous knapsack problem, our intuition tells us that we may order variables based on the ratios. If you do so and find the correct optimal solution, you will get full points. Nevertheless, below we show that the original problem is indeed equivalent to the continuous knapsack problem. First, let $y_i = 1 - x_i$ for $i = 1, ..., 4$, we obtain

$$\begin{aligned} \min \quad & 18 - 3x_1 - 7x_2 - 2x_3 - 6x_4 \\ \text{s.t.} \quad & 13 - 4x_1 - 3x_2 - 4x_3 - 2x_4 \geq 7 \\ & 0 \leq x_j \leq 1 \quad \forall j = 1, ..., 4, \end{aligned}$$

which is equivalent to the following continuous knapsack problem

$$\begin{aligned} \max \quad & 3x_1 + 7x_2 + 2x_3 + 6x_4 \\ \text{s.t.} \quad & 4x_1 + 3x_2 + 4x_3 + 2x_4 \leq 6 \\ & 0 \leq x_j \leq 1 \quad \forall j = 1, ..., 4. \end{aligned}$$

To solve this problem, we follow the ratio rule and first choose x_4 to be 1. As the remaining capacity $4 > 0$, we choose x_2 to be 1. As the remaining capacity $1 > 0$, we choose x_1 to be $\frac{1}{4}$. x_3 is then set to be 0. The optimal solution to the original program is then found through $y_i = 1 - x_i$ as $(y_1^*, y_2^*, y_3^*, y_4^*) = (0, \frac{3}{4}, 1, 0)$.

(d) To solve this problem, first we recognize that the feasible region is convex. Moreover, we find that we are minimizing a convex function. This implies that there exists an extreme point optimal solution. As $y_4 = 0$, the remaining constraints tell us that there are eight extreme points, which satisfy $y_i \in \{0, 1\}$ for $i = 1, ..., 3$. All we need to do is to evaluate these eight candidates and pick up the one with the highest objective value. The calculation is summarized below:

y_1	y_2	y_3	$f(y)$
0	0	0	0
0	0	1	2.29
0	1	0	2.99
0	1	1	1.71
1	0	0	4.25
1	0	1	2.94
1	1	0	3.67
1	1	1	1.76

Therefore, the optimal solution is $(y_1^*, y_2^*, y_3^*, y_4^*) = (1, 0, 0, 0)$.

(e) This program can be separated into four independent problems, one for each variable. For y_1, this is a convex program and FOC suggests the solution $\bar{y}_1 = -1$. As it is infeasible, the optimal solution is $y_1^* = 0$. For y_2, this is also a convex program and FOC results in the optimal solution $y_2^* = \frac{1}{4}$. For y_3, because e^{2y_3} is increasing in y_3, the optimal solution is $y_3^* = 1$. For y_4, because $\sqrt{y_4 + 3}$ is increasing in y_4, the optimal solution is $y_4^* = 1$. Collectively, the optimal solution to the original program is $(y_1^*, y_2^*, y_3^*, y_4^*) = (0, \frac{1}{4}, 1, 1)$.

3. (a) The EOQ is $\sqrt{\frac{2(20)(1000)}{1}} = 200$.

(b) The EOQ is $\sqrt{\frac{2(5)(2000)}{2}} = 100$.

(c) The order cycle time of these two products are $\frac{200}{1000} = 0.2$ year and $\frac{100}{2000} = 0.05$ year, respectively. Suppose we follow the EOQ rule to order product 1 and obtain 200 units at the beginning of a year. If we also want to follow the EOQ rule for product 2, to best utilize our warehouse, we should order product 2 after 0.025 year. However, because in 0.025 years we only consume $1000 \times 0.025 = 25$ units of product 2, we only have a room of $(250-200)+25 = 75$ units in the warehouse, which is not enough for the 100 units in an order of product 2. Therefore, we cannot follow the EOQ rule for both products.

Note. If you say no only because $200 + 100 > 250$, you will only get partial credits.

4. (a) First, we define $V = \{1, ..., n\}$ and $V_{i+} = \{i+1, ..., n\}$. Let

$$x_i = \begin{cases} 1 & \text{if there is an office in city } i \\ 0 & \text{otherwise} \end{cases}, i \in V, \text{ and}$$

$$y_{ij} = \begin{cases} 1 & \text{if there is a direct flight between cities } i \text{ and } j, \\ 0 & \text{otherwise} \end{cases}, i \in V, j \in V_{i+},$$

be the decision variables. The formulation is

$$\begin{aligned} \max \quad & \sum_{i \in V} \sum_{j \in V_{i+}} (R_{ij} - C_{ij}) y_{ij} - \sum_{i \in V} F_i x_i \\ \text{s.t.} \quad & y_{ij} \leq x_i \quad \forall i \in V, j \in V_{i+} \\ & y_{ij} \leq x_j \quad \forall i \in V, j \in V_{i+} \\ & x_i,\ y_{ij} \in \{0, 1\} \quad \forall i \in V, j \in V_{i+}. \end{aligned}$$

In particular, the first two constraints ensure that we may operate a direct flight between cities i and j only if we operate offices in both cities i and j.

(b) Continue from Part (a), we define two more sets of decision variables

$$z_{ij} = \begin{cases} 1 & \text{if there are offices both in cities } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}, i \in V, j \in V_{i+}$$

$$w_{ij} = \begin{cases} 1 & \text{if there is a one-stop indirect flight but no direct flight between cities } i \text{ and } j \\ 0 & \text{otherwise} \end{cases},$$

$i \in V, j \in V_{i+}$

With these new variables, we add the following term

$$\alpha \sum_{i \in V} \sum_{j \in V_{i+}} R_{ij} w_{ij},$$

which represents the revenue collected from indirect flights, into the objective function. Now the question is how to correctly connect w_{ij} with y_{ij} and z_{ij}. First, we add the following constraint

$$2z_{ij} \leq y_{ik} + y_{kj} \quad \forall i \in V, j \in V_{i+}, k \in V \setminus \{i, j\}$$

to connect z_{ij} and y_{ij}: If there is a city $k \notin \{i, j\}$ such that $y_{ik} = y_{kj} = 1$, i.e., there are direct flights between i and k and between k and j, we know there is a one-stop flight between i and j and thus z_{ij} can be 1. We cannot use z_{ij} to determine the collection of revenues from indirect flights because even if $z_{ij} = 1$, it is still possible that there is a direct flight between i and j. Therefore, we introduce w_{ij} and add the following constraint

$$w_{ij} \leq z_{ij} - y_{ij} \quad \forall i \in V, j \in V_{i+}$$

to ensure that $w_{ij} = 1$ only if $z_{ij} = 1$ and $y_{ij} = 0$. Certainly we also need to add the binary constraints $z_{ij}, w_{ij} \in \{0, 1\}$ for all $i \in V, j \in V_{i+}$.

(c) For the linear relaxation, an optimal solution is not always an integer solution. To see this, consider a two-city instance in which the cost of operating an office in each city is 5 and the net benefit of operating the flight between the two cities is 10. Let the two cities be cities 1 and 2. While one optimal solution is $x_1 = x_2 = y_{12} = 1$, another optimal solution is $x_1 = x_2 = y_{12} = 0$. Moreover, $x_1 = x_2 = y_{12} = h$ for any $h \in (0, 1)$ is also an optimal solution, which implies an optimal solution may not be an integer solution.

Note. In fact, for the integer program in Part (a), relaxing the binary constraints is fine, i.e., there is always an integer optimal solution to the linear relaxation (even through there may be some non-integer ones) and the simplex method will always find an integer optimal solution. To prove this, we may either show that the program in Part (a) is *totally unimodular* or it is equivalent to a *maximum flow* problem. These are certainly beyond the scope of this course.

5. There are two types of bidders: The one whose valuation is the highest and those whose valuations are not the highest. We will discuss these two cases separately and show that no one will unilaterally deviate from the strategy "bidding my valuation" in either case:

   - Suppose I have the highest valuation and all other bidders bid at their valuations. In this case, if I bid at my valuation, I will win the item and pay the second highest bid, which is the second highest valuation. Therefore, I can get a positive valuation. If I bid a higher price, the expected utility does not increase: I will still win the item by paying the same price. If I bid a lower price, however, the expected utility decreases because I may lose the item. Note that even if I bid a lower price that still allows me to win the item, my utility does not become higher because I still pay the same amount, the second highest bid, to the seller.
   - Suppose I does not have the highest valuation and all other bidders bid at their valuations. In this case, if I bid my valuation, I will not win the item and my utility will be zero. Suppose I bid a lower price, my expected utility does not increase: It is still zero because I will still not win the item. Suppose I bid a higher price, however, my expected utility decreases. This is because I may win the item by paying the second highest bid, which is the highest valuation. As the highest valuation is higher than my valuation, my utility will be negative once I win the item.

   Therefore, we conclude that as long as all other bidders bid at their valuation, I have no incentive to deviate from bidding at my valuation. That all bidders bid at their valuations is thus a Nash equilibrium.

6. (a) To show that there is no saddle point, first we find those row minima and column maxima:

	E	F	row min
A	2	-2	-2
B	1	-1	-1
C	0	0	0
D	-1	2	-1
column max	2	2	

   The minimum of column max, 2, is not identical to the maximum of row min, 0. Therefore, there is no saddle point.

   (b) Let x_1, x_2, x_3, and x_4 be the probabilities for player 1 to select A, B, C, and D, respectively. Player 1's problem can then be formulated as

$$\begin{aligned} \max \quad & u \\ \text{s.t.} \quad & u \leq 2x_1 + x_2 - x_4 \\ & u \leq -2x_1 - x_2 + 2x_4 \\ & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_i \geq 0 \quad \forall i = 1, ..., 4. \end{aligned}$$

   (c) The first way of finding player 1's equilibrium strategy is to solve the linear program in Part (b) directly. However, as there are five variables with three constraints, running the simplex method may require a lot of time. We may choose another way by utilizing duality. First, the dual linear program (which is player 2's problem with y_1 and y_2 being the probabilities of choosing E and F, respectively) is

$$\begin{aligned} \min \quad & v \\ \text{s.t.} \quad & v \geq 2y_1 - 2y_2 \\ & v \geq y_1 - y_2 \\ & v \geq 0 \\ & v \geq -y_1 + 2y_2 \\ & y_1 + y_2 = 1 \\ & y_i \geq 0 \quad \forall i = 1, 2. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min \quad & v \\ \text{s.t.} \quad & v \geq -2 + 4y_1 \\ & v \geq -1 + 2y_1 \\ & v \geq 0 \\ & v \geq 2 - 3y_1 \\ & 0 \leq y_1 \leq 1, \end{aligned}$$

where the equivalence is based on $y_2 = 1 - y_1$. The latter can be solved graphically as in Figure 1. As we find that constraints 2 and 3 are nonbinding, at an primal optimal solution x_2 and x_3 must be 0. Therefore, the primal program reduces to

$$\begin{array}{ll} \max & u \\ \text{s.t.} & u \leq 2x_1 - x_4 \\ & u \leq -2x_1 + 2x_4 \\ & x_1 + x_4 = 1 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \max & u \\ \text{s.t.} & u \leq -1 + 3x_1 \\ & u \leq 2 - 4x_1 \\ & 0 \leq x_1 \leq 1, \end{array}$$

where the equivalence is based on $x_2 = 1 - x_1$. The latter can be solved graphically as in Figure 2 with $x_1^* = \frac{3}{7}$ being a part of an optimal solution. This implies that an optimal solution to the original player 1's problem is $(x_1^*, x_2^*, x_3^*, x_4^*) = (\frac{3}{7}, 0, 0, \frac{4}{7})$.

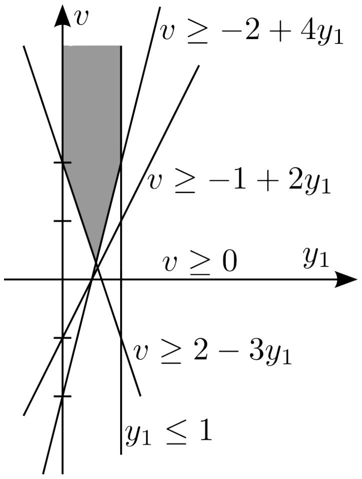

Figure 1: Solving the dual for Problem 6c.

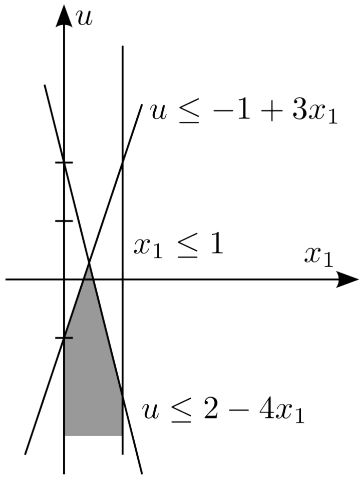

Figure 2: Solving the primal for Problem 6c.

7. (a) Let $\pi_R(q)$ be the retailer's expected profit when ordering q units, we have

$$\begin{aligned} \pi_R(q) = r\mathbb{E}\min\{q, D\} - wq &= \left\{ \int_0^q x\left(\frac{1}{b}\right)dx + \int_q^b q\left(\frac{1}{b}\right)dx \right\} - wq \\ &= \frac{r}{b}\left[\frac{1}{2}q^2 + q(b-q)\right] - wq = -\frac{4}{2b}q^2 + (r-w)q \end{aligned}$$

and the retailer's problem is to solve $\max_{q \geq 0} \pi_R(q)$. By applying the newsvendor formula, we have

$$1 - F(q^*(w)) = \frac{w}{r} \quad \Rightarrow \quad 1 - \frac{q^*(w)}{b} = \frac{w}{r} \quad \Rightarrow \quad q^*(w) = \left(\frac{r-w}{r}\right)b.$$

Note. To get full credits for formulation, you do not need to do the Calculus. Those derivations are only for calculating the retailer's equilibrium expected profit in Part (c).

(b) Let $\pi_M(w)$ be the manufacturer's profit when choosing the wholesale price w, we have

$$\pi_M(w) = (w-c)q^*(w) = (w-c)\left(\frac{r-w}{r}\right)b$$

and the manufacturer's problem is $\max_{w \geq c} \pi_M(w)$. By applying the FOC, we have $w^* = \frac{r+c}{2}$.

(c) The retailer's equilibrium order quantity

$$q^* = q^*(w^*) = \left(\frac{r-c}{2r}\right)b.$$

The retailer's equilibrium expected profit is

$$\pi_R^* = \pi_R(q^*) = -\frac{r}{2b}\left[\frac{(r-c)^2}{4r^2}\right]b^2 + \left(\frac{r-c}{2}\right)\left(\frac{r-c}{2r}\right)b = \frac{b(r-c)^2}{8r}.$$

The manufacturer's equilibrium profit is

$$\pi_M^* = \pi_M(w^*) = \left(\frac{r-c}{2}\right)\left(\frac{r-c}{2r}\right)b = \frac{b(r-c)^2}{4r}.$$

(d) Suppose the manufacturer faces consumers directly, she solves

$$\max_{Q\geq 0} = r\mathbb{E}\min\{Q, D\} - cQ,$$

whose optimal solution (through the newsvendor formula) is $Q^* = (\frac{r-c}{r})b$. As $q^* = \frac{r-c}{2r}b$, it is clear that $Q^* > q^*$.

(e) For any continuous random variable D with cdf F, the newsvendor formula implies that

$$F(q^*) = 1 - \frac{w^*}{r} < 1 - \frac{c}{r} = F(Q^*),$$

where the inequality comes from the fact that the equilibrium wholesale price w^* must be higher than the unit production cost c. As F is an increasing function, we have $Q^* > q^*$.