Operations Research, Spring 2013
Suggested Solution for Midterm Exam

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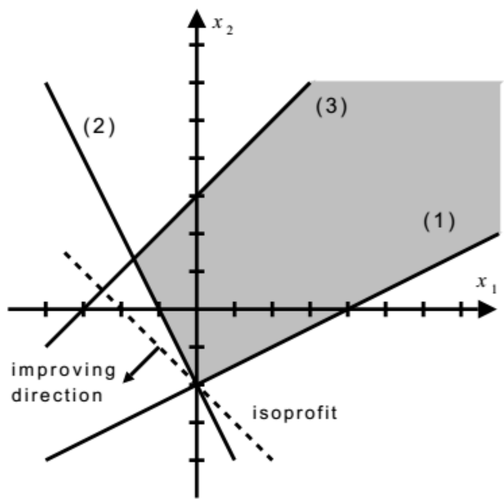
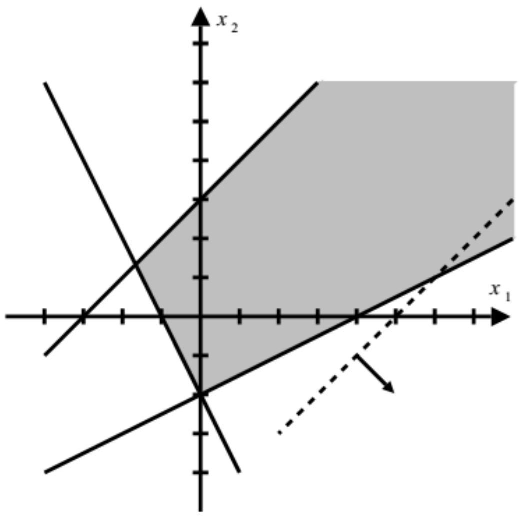
1. (a) False. The correct statement is "If a standard form linear program has an optimal solution, then it has an optimal basic feasible solution."
   (b) True. Any solution that is a convex combination of the two distinct optimal solutions is an optimal solution.
   (c) True. By strong duality.
   (d) False. This statement is true if and only if the two solutions are primal and dual feasible.
   (e) False. The dual of (D) is (P), which is unbounded.

2. (a) The optimal solution is $(x_1, x_2) = (0, -2)$ (see the graph below).



   (b) The problem is unbounded (see the graph below). No matter how far you push your isoprofit line, you can still find a feasible point on the isoprofit line.



3. (a) The standard form is

$$\begin{aligned} \max \quad & x_1 + x_2' \\ \text{s.t.} \quad & 2x_1 + x_2' - x_3 = 4 \\ & x_1 + x_2' + x_4 = 3 \\ & x_1, x_2', x_3, x_4 \geq 0. \end{aligned}$$

(b) The six basic solutions are listed in the following table. Three of them are feasible.

x_1	x_2'	x_3	x_4	Feasible?
0	0	−4	3	No
0	4	0	−1	No
0	3	−1	0	No
2	0	0	1	Yes
3	0	2	0	Yes
1	2	0	0	Yes

(c) The current tableau is optimal since all numbers in the objective row is nonnegative. The current optimal solution is $(x_1^*, x_2^*) = (1, -2)$ (for the original variables). Since there is one nonbasic column (the third column) having 0 in the objective row, we enter x_3:

$$\begin{array}{cccc|c} 0 & 0 & 0 & 1 & 3 \\ \hline 1 & 0 & -1 & -1 & 1 \\ 0 & 1 & \boxed{1} & 2 & 2 \end{array} \quad \rightarrow \quad \begin{array}{cccc|c} 0 & 0 & 0 & 1 & 3 \\ \hline 1 & 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 & 2 \end{array}$$

This new tableau gives us another optimal solution $(x_1^{**}, x_2^{**}) = (3, 0)$.

4. (a) The standard form is

$$\begin{array}{lcccccccccccl} \max & x_1 & - & 2x_2 & & & & & & & & & \\ \text{s.t.} & x_1 & & & - & 2x_3 & + & x_4 & & & & = & 6 \\ & x_1 & - & 3x_2 & + & x_3 & & & + & x_5 & & = & 12 \\ & & & 3x_2 & - & 2x_3 & & & & & + \ x_6 & = & 8 \\ & & & & & & & & & & x_i & \geq & 0 \quad \forall\, i = 1, ..., 6. \end{array}$$

We run two iterations to get the optimal tableau:

$$\begin{array}{cccccc|c} -1 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline \boxed{1} & 0 & -2 & 1 & 0 & 0 & x_4 = 6 \\ 1 & -3 & 1 & 0 & 1 & 0 & x_5 = 12 \\ 0 & 3 & -2 & 0 & 0 & 1 & x_6 = 8 \end{array} \rightarrow \begin{array}{cccccc|c} 0 & 2 & -2 & 1 & 0 & 0 & 0 \\ \hline 1 & 0 & -2 & 1 & 0 & 0 & x_1 = 6 \\ 0 & -3 & \boxed{3} & -1 & 1 & 0 & x_5 = 6 \\ 0 & 3 & -2 & 0 & 0 & 1 & x_6 = 8 \end{array}$$

$$\rightarrow \begin{array}{cccccc|c} 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 10 \\ \hline 1 & -2 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & x_1 = 10 \\ 0 & -1 & 1 & -\frac{1}{3} & \frac{1}{3} & 0 & x_3 = 2 \\ 0 & 1 & 0 & -\frac{2}{3} & \frac{2}{3} & 1 & x_6 = 12 \end{array}$$

The current basic feasible solution corresponds to an optimal solution $x^* = (x_1^*, x_2^*, x_3^*) = (10, 0, 2)$ for the original problem. Because x_2 is nonbasic and has 0 reduced cost, there are multiple optimal solutions.

5. First we write down the Phase-I LP

$$\begin{aligned} \max \quad & -x_3 \\ \text{s.t.} \quad & x_1 - x_2 + x_3 = 2 \\ & x_1 + x_4 = 4 \\ & x_j \geq 0 \quad \forall\, j = 1, ..., 4. \end{aligned}$$

Here x_3 is an artificial variable and x_4 is a slack variable. An initial basic feasible solution for the augmented form is $(0, 0, 2, 4)$. Because

$$\begin{array}{cccc|c} 0 & 0 & 1 & 0 & 0 \\ \hline 1 & -1 & 1 & 0 & x_3 = 2 \\ 1 & 0 & 0 & 1 & x_4 = 4 \end{array}$$

is not a valid tableau, we fix the objective row and then run one simplex iteration:

$$\begin{array}{cccc|c} -1 & 1 & 0 & 0 & -2 \\ \hline \boxed{1} & -1 & 1 & 0 & x_3 = 2 \\ 1 & 0 & 0 & 1 & x_4 = 4 \end{array} \quad \rightarrow \quad \begin{array}{ccc|c} 0 & 0 & 0 & 2 \\ \hline 1 & -1 & 0 & x_1 = 2 \\ 0 & 1 & 1 & x_4 = 2 \end{array}$$

We delete the column for x_3 when x_3, the artificial variable, leaves the basis. This gives us a basic feasible solution $(x_1, x_2, x_4) = (2, 0, 2)$ to the standard form LP (with only x_1, x_2, and x_4). We then put the original objective function back:

$$\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & x_1 = 2 \\ 0 & 1 & 1 & x_4 = 2 \end{array}$$

Again, we fix the objective row and do one more iteration to find the optimal solution:

$$\begin{array}{ccc|c} 0 & -1 & 0 & 2 \\ \hline 1 & -1 & 0 & x_1 = 2 \\ 0 & \boxed{1} & 1 & x_4 = 2 \end{array} \quad \rightarrow \quad \begin{array}{ccc|c} 0 & 0 & 1 & 4 \\ \hline 1 & 0 & 1 & x_1 = 4 \\ 0 & 1 & 1 & x_2 = 2 \end{array}$$

Therefore, the original program has a unique optimal solution $(x_1, x_2) = (4, 2)$ with the objective value 4.

6. **Formulation 1.** We label the bundle as product 4. Let

$$x_i = \text{sales quantity of product } i,\ i = 1, ..., 4.$$

Define $S = (500, 500)$ as the supply vector, $P = (20, 30, 15, 45)$ as the price vector, $D = (100, 50, 120)$ as the demand vector, and

$$R = \begin{bmatrix} 2 & 1 \\ 3 & 0 \\ 1 & 2 \\ 5 & 1 \end{bmatrix}$$

as the material consumption matrix. The problem can then be formulated as

$$\begin{aligned} \max \quad & \textstyle\sum_{i=1}^{4} P_i x_i & & \text{(Maximize total sales revenue)} \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{4} R_{ij} x_i \leq S_j \quad \forall\, j = 1, 2 & & \text{(Supply limitation)} \\ & x_i \leq D_i \quad \forall\, i = 1, ..., 3 & & \text{(Demand limitation)} \\ & x_i \geq 0 \quad \forall\, i = 1, ..., 4. \end{aligned}$$

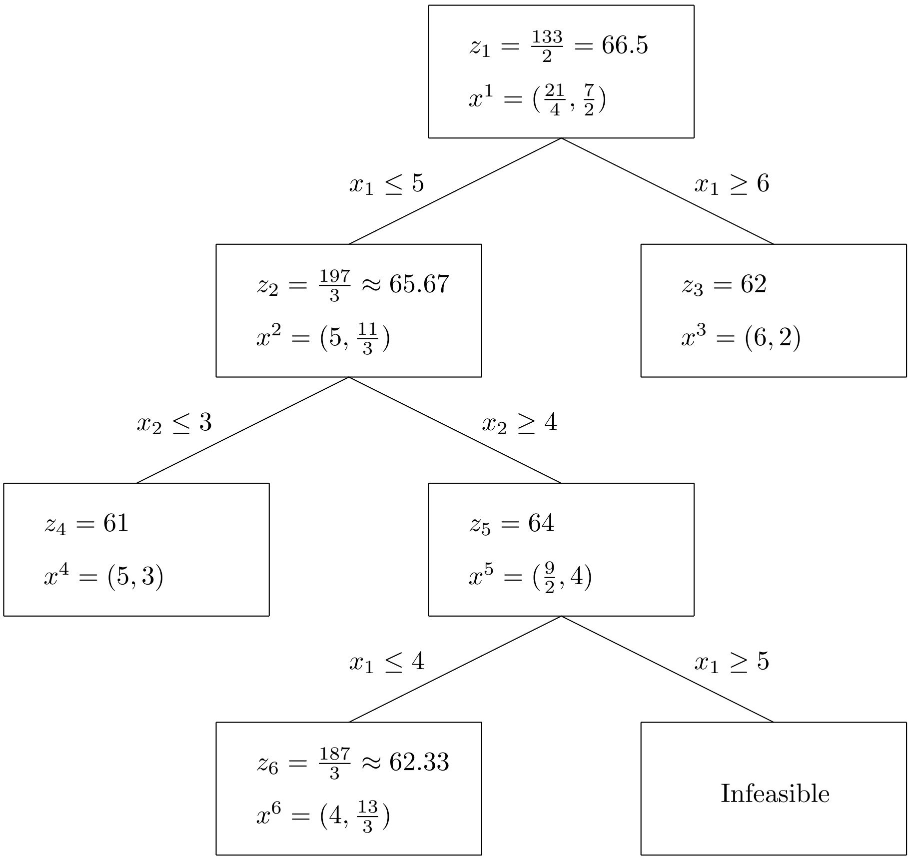
Formulation 2. Alternatively, we may formulate this problem in another way. Let

$$y_i = \text{production quantity of product } i,\ i = 1, ..., 3 \text{ and}$$
$$y_4 = \text{sales quantity of product 4.}$$

The problem is then formulated as

$$\begin{array}{lll} \max & \sum_{i=1}^{2} P_i(y_i - y_4) + \sum_{i=3}^{4} P_i y_i & \text{(Maximize total sales revenue)} \\ \text{s.t.} & \sum_{i=1}^{3} R_{ij} x_i \leq S_j \quad \forall\, j = 1, 2 & \text{(Supply limitation for products 1, ..., 3)} \\ & y_4 \leq y_1, \quad y_4 \leq y_2 & \text{(Supply limitation for product 4)} \\ & y_i - y_4 \leq D_i \quad \forall\, i = 1, 2 & \text{(Demand limitation for products 1 and 2)} \\ & y_3 \leq D_3 & \text{(Demand limitation for product 3)} \\ & y_i \geq 0 \quad \forall\, i = 1, ..., 4. & \end{array}$$

7. A branch-and-bound tree is depicted below. For each node, we solve a two-dimensional problem by the graphical approach. The optimal solution for the IP is $x^* = (x_1^*, x_2^*) = (6, 2)$ and the corresponding objective value is $z^* = 62$.



8. (a) Let

$$x_{ij} = \text{number of consumers served in city } j \text{ of county } i,\ i = 1, ..., 10,\ j = 1, ..., 5,$$
$$y_i = \begin{cases} 1 & \text{if a DC is built in county } i \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., 10,$$
$$z_{ij} = \begin{cases} 1 & \text{if a store is built in city } j \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., 10, j = 1, ..., 5,$$

be the decision variables. The problem can then be formulated as

$$\begin{aligned}
\max \quad & \sum_{i=1}^{10}\sum_{j=1}^{5}\Big[(R_{ij}-C_{ij})x_{ij}-F_{ij}z_{ij}\Big]-\sum_{i=1}^{10}H_iy_i \\
\text{s.t.} \quad & \sum_{j=1}^{5} x_{ij} \le K_i y_i \quad \forall\, i=1,...,10 \\
& x_{ij} \le D_{ij}z_{ij} \quad \forall\, i=1,...,10, j=1,...,5 \\
& x_{ij} \ge 0, z_{ij} \in \{0,1\} \quad \forall\, i=1,...,10, j=1,...,5 \\
& y_i, w_i \in \{0,1\} \quad \forall\, j=1,...,5.
\end{aligned}$$

The objective function maximizes the total sales revenue minus the total cost (unit cost and construction costs). The first constraint ensures that we do not serve more consumers than we can and relates serving consumers and building DCs: When we serve any consumer in county i ($x_{ij} > 0$ for any j), we need to build a DC ($y_i = 1$). The second constraint ensure that we do not server more consumer than we have and relates serving consumers and building stores: When we serve any consumer in city j of county i ($x_{ij} > 0$), we need to build a store ($z_{ij} = 1$). The other constraints ensure variables are nonnegative or binary.

(b) $\sum_{i=3}^{4}\sum_{j=1}^{5} z_{ij} \ge 3$.

(c) Let a_i be 1 is there are at least three stores in county i and 0 otherwise, $i = 1, ..., 10$. Then we add the following constraint $\sum_{j=1}^{5} z_{ij} \ge 3a_i$ for all $i = 1, ..., 10$ to ensure that if we want to have $a_i = 1$, we need to build at least three stores in county i. This and the constraint $\sum_{i=1}^{10} a_i \ge 5$ together meets our requirement.

9. Let the basis be $B = \{x_3, x_1\}$ and the nonbasic variables be $N = \{x_2, x_4, x_5\}$, we have

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad B^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}, \quad \text{and } c_B = \begin{bmatrix} 1 & 3 \end{bmatrix}.$$

(a) We have

$$c_B B^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{5}{3} \end{bmatrix},$$

which implies that the shadow prices for constraints 1 and 2 are $-\frac{1}{3}$ and $\frac{5}{3}$, respectively.

(b) The dual LP is

$$\begin{array}{rrcrcr}
\max & 6y_1 & + & 9y_2 & & \\
\text{s.t.} & y_1 & + & 2y_2 & \ge & 3 \\
& y_1 & + & 2y_2 & \ge & 2 \\
& 2y_1 & + & y_2 & \ge & 1 \\
& \multicolumn{5}{l}{y_1 \le 0, y_2 \ge 0.}
\end{array}$$

(c) Suppose dual constraint 1 is nonbinding at a dual optimal solution, its shadow price, which is the value of x_1 in the primal optimal solution, must be 0. As this is not true, dual constraint 1 must be binding. Similarly, dual constraint 3 must also be binding. We may then solve the linear system

$$\begin{aligned} y_1 + 2y_2 &= 3 \\ 2y_1 + y_2 &= 1 \end{aligned}$$

and obtain a dual optimal solution $(y_1^*, y_2^*) = (-\frac{1}{3}, \frac{5}{3})$. As this is identical to the solution we find in Part (a), the primal shadow prices are verified.