# Operations Research, Spring 2013
# Homework 01

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1. First we represent this linear system in the augmented matrix form

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 4 \\ 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 \end{array}\right].$$

Then we apply the Gauss-Jordan elimination as follows:

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 4 \\ 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

With this we know the unique solution is $(x_1, x_2, x_3) = (1, 1, 1)$.

2. We use the Gauss-Jordan elimination to compute the inverse:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 4 & 1 & -2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 1 & -4 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right]$$
$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -6 & -4 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & -2 & 3 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right].$$

Therefore, the inverse is

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & -2 & 3 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

3. (a) This is not a convex set. For example, the line segment connecting two feasible solutions -3 and 3 does not completely lie in this set.
   (b) This is a convex set as it is a feasible region of a linear program.
   (c) This is not a convex set. For example, the line segment connecting two feasible solutions $(1, 0)$, $(e, 1)$ does not completely lie in this set.
   (d) This is a convex set. The first constraint $\frac{x^2}{9} + \frac{y^2}{4} \leq 1$ results in a feasible region that is the area inside an ellipse. The other constraints are all linear and result in half planes. As all these sets are convex, the intersection is also convex.
   (e) This is a convex set as it is a feasible region of a linear program.

4. (a) This is a convex function: all line segments connecting two points on the curve are above the curve.
   (b) This is not a convex function. For example, the line segment connecting the two points $(-1, -1)$ and $(0, 0)$ on the curve is below the curve.

(c) This is not a convex function. For example, the line segment connecting the two points $(2, 1)$ and $(5, 10)$ on the curve is not completely weakly above the curve.

(d) This is not a convex function. For example, the line segment connecting the two points $(-1, 1)$ and $(0, 0)$ on the curve is below the curve.

(e) This is a convex function as it is a linear function.

5. (a) We have

$$A = \begin{bmatrix} 6 & 8 & -4 \\ -5 & 3 & 2 \\ 1 & 0 & 2 \\ -1 & 0 & -2 \\ -1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ -7 \\ 4 \\ -4 \\ 0 \end{bmatrix}, \text{and} \quad c = \begin{bmatrix} -1 & -2 & 3 \end{bmatrix}.$$

(b) $(x_1, x_2, x_3) = (2, 0, 1)$ is not feasible as it violates $5x_1 - 3x_2 - 2x_3 \geq 7$. The corresponding objective value is $1 \times 2 + 2 \times 0 - 3 \times 1 = -1$.

(c) $(x_1, x_2, x_3) = (1, 1, 2)$ is not feasible as it violates $x_1 + 2x_3 = 4$. The corresponding objective value is $1 \times 1 + 2 \times 1 - 3 \times 2 = -3$.

**Note.** It is fine if you calculate the objective values with the minimization objective function you find in Part (a).

6. Let

$$x_1 = \text{number of acres of corn planted and}$$
$$x_2 = \text{number of acres of wheat planted.}$$

Then the problem can be formulated as

$$\begin{array}{rrcrcll}
z^* = \max & 30x_1 & + & 100x_2 & & & \\
\text{s.t.} & 4x_1 & + & 10x_2 & \leq & 40 & \text{(Total amount of labor hours)} \\
 & x_1 & + & x_2 & \leq & 7 & \text{(Total amount of land)} \\
 & 10x_1 & & & \geq & 30 & \text{(Minimum quantity of corn produced)} \\
 & x_1 & & & \geq & 0 & \text{(Not required since we already have } 10x_1 \geq 30) \\
 & & & x_2 & \geq & 0. &
\end{array}$$

7. The decision variables are

$$x_i = \text{number of valves purchased from supplier } i,\ i = 1, ..., 3.$$

♦ **Solution 1** The problem can be formulated as

$$\begin{array}{rrcrcrcll}
\min & 5x_1 & + & 4x_2 & + & 3x_3 & & & \\
\text{s.t.} & 0.4x_1 & + & 0.3x_2 & + & 0.2x_3 & \geq & 500 & \text{(Demand for large valves)} \\
 & 0.4x_1 & + & 0.35x_2 & + & 0.2x_3 & \geq & 300 & \text{(Demand for medium valves)} \\
 & 0.2x_1 & + & 0.35x_2 & + & 0.6x_3 & \geq & 300 & \text{(Demand for small valves)} \\
 & x_1 & & & & & \leq & 700 & \text{(Supply from supplier 1)} \\
 & & & x_2 & & & \leq & 700 & \text{(Supply from supplier 2)} \\
 & & & & & x_3 & \leq & 700 & \text{(Supply from supplier 3)} \\
 & x_1 & & & & & \geq & 0 & \\
 & & & x_2 & & & \geq & 0 & \\
 & & & & & x_3 & \geq & 0. &
\end{array}$$

♦ **Solution 2** As the problems become more and more complicated, we will use a more concise way to write our formulation. For this problem, we label large, medium, and small valves by valve 1, 2, and 3. We then define the following parameters: $C_i$ is the unit purchasing cost of supplier $i$, $D_j$ is the demand quantity of valve $j$, $S_i$ is the supply limit of supplier $i$, and $P_{ij}$ is the percentage of valve $j$ of valves purchased from supplier $i$. Still with $x_i$ being the number of valves purchased from supplier $i$ ($i = 1, ..., 3$), we formulate the problem as

$$\begin{aligned}
\min \quad & \sum_{i=1}^{3} C_i x_i \\
\text{s.t.} \quad & \sum_{i=1}^{3} P_{ij} x_i \geq D_j \quad \forall\, j = 1, 2, 3 \\
& x_i \leq S_i \quad \forall\, i = 1, 2, 3 \\
& x_i \geq 0 \quad \forall\, i = 1, 2, 3.
\end{aligned}$$

**Remark.** Please compare the two formulations and understand the compact one. Typically we use capital letters for parameters and small letters for variables. This is good for you to distinguish parameters and constraints and avoid nonlinear formulations.

8. To formulate this problem, we label the six shifts as in the following table.

| Shift Number | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Time | 0 - 4 | 4 - 8 | 8 - 12 | 12 - 16 | 16 - 20 | 20 - 24 |

The decision variables are

$$x_i = \text{number of officers starting to work at shift } i,\ i = 1, ..., 6.$$

We also define $D_i$ as the number of officers required for shift $i$ for $i = 1, ..., 6$. Specifically, we have

$$D = \begin{bmatrix} 8 & 7 & 6 & 6 & 5 & 4 \end{bmatrix}.$$

The problem can then be formulated as

$$\begin{aligned}
\min \quad & \sum_{i=1}^{6} x_i \\
\text{s.t.} \quad & x_1 + x_6 \geq D_1 \\
& x_i + x_{i+1} \geq D_{i+1} \quad \forall\, i = 1, ..., 5 \\
& x_i \geq 0 \quad \forall\, i = 1, ..., 6.
\end{aligned}$$

9. Let the decision variables be

$$x_{ij} = \text{oz of chemical } i \text{ used for producing drug } j,\ i = 1, 2,\ j = 1, 2.$$

Also we define the following parameters: $P_j$ is the sales price of one oz of drug $j$, $C_i$ is the purchasing cost of one oz of chemical $i$, $S_i$ is the total amount of supply of chemical $i$ (in oz), and $D_j$ is the demand size of drug $j$ (in oz) for $i = 1, 2$ and $j = 1, 2$. Specifically, we have

$$P = \begin{bmatrix} 6 & 5 \end{bmatrix}, \quad C = \begin{bmatrix} 6 \\ 4 \end{bmatrix}, \quad S = \begin{bmatrix} 45 \\ 40 \end{bmatrix}, \quad D = \begin{bmatrix} 40 & 30 \end{bmatrix}.$$

With the definitions of variables and parameters, we formulate the problem as

$$\begin{aligned}
\max \quad & \sum_{j=1}^{2} P_j \sum_{i=1}^{2} x_{ij} - \sum_{i=1}^{2} C_i \sum_{j=1}^{2} x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{2} x_{ij} \leq D_j \quad \forall\, j = 1, 2 \\
& \sum_{j=1}^{2} x_{ij} \leq S_i \quad \forall\, i = 1, 2 \\
& 0.3x_{11} - 0.7x_{21} \geq 0, \quad -0.6x_{12} + 0.4x_{22} \geq 0 \\
& x_{ij} \geq 0 \quad \forall\, i = 1, 2, j = 1, 2.
\end{aligned}$$

The objective function consists of two parts, the sales revenue and the purchasing cost. The first constraint ensures that the total sales of each drug point is at most the demand size. The second constraint ensures that the total usage of each chemical does not excess the supply quantity. The third constraint ensures the quality. *It is very important that you do not use a nonlinear constraint to ensure quality.* The last constraint is the nonnegativity constraint.

10. First we label the following four months as months 1, 2, 3, and 4. Then let the decision variables be

$$\begin{aligned} x_t &= \text{ending inventory of month } t,\ t = 0, ..., 4, \text{ and} \\ y_t &= \text{production quantity in month } t,\ t = 1, ..., 4. \end{aligned}$$

Also we define the following parameters: $D_t$ is the demand size of month $t$ and $C_t$ is the unit production cost in month $t$ for all $t = 1, ..., 4$. Specifically, we have

$$D = \begin{bmatrix} 50 & 65 & 100 & 70 \end{bmatrix}, \quad C = \begin{bmatrix} 5 & 8 & 4 & 7 \end{bmatrix}.$$

With the definitions of variables and parameters, we formulate the problem as

$$\begin{aligned} \min \quad & \sum_{t=1}^{4} C_t y_t + 2\sum_{t=1}^{4} x_t - 6x_4 \\ \text{s.t.} \quad & x_{t-1} + y_t - D_t = x_t \quad \forall\, t = 1, ..., 4 \\ & x_0 = 0 \\ & x_t, y_t \geq 0 \quad \forall\, 0 = 1, ..., 4. \end{aligned}$$

The objective function minimizes the net cost. The first constraint is the inventory balancing constraint. The second constraint sets the initial inventory to 0. The last constraint is the nonnegativity constraint.

11. First we label cheesecakes as product 1 and black forest cakes as product 2. Let the decision variables be

$$\begin{aligned} x_{it} &= \text{ending inventory of product } i \text{ in month } t,\ i = 1, 2,\ t = 0, ..., 3, \text{ and} \\ y_{it} &= \text{production quantity of product } i \text{ in month } t,\ i = 1, 2,\ t = 1, ..., 3. \end{aligned}$$

Also we define the following parameters: $K$ is the monthly capacity for producing both products, $D_{it}$ is the demand size of product $i$ in month $t$, $C_{it}$ is the unit production cost of product $i$ in month $t$, and $H_i$ is the per month unit holding cost of product $i$ for $i = 1, 2$ and $t = 1, ..., 3$. Specifically, we have

$$K = 65, \quad D = \begin{bmatrix} 40 & 30 & 20 \\ 20 & 30 & 10 \end{bmatrix}, \quad C = \begin{bmatrix} 3.00 & 3.40 & 3.80 \\ 2.50 & 2.80 & 3.40 \end{bmatrix}, \quad H = \begin{bmatrix} 0.50 \\ 0.40 \end{bmatrix}.$$

With the definitions of variables and parameters, we formulate the problem as

$$\begin{aligned} \min \quad & \sum_{i=1}^{2} \left( \sum_{t=1}^{3} C_{it} y_{it} + H_i \sum_{t=1}^{3} x_{it} \right) \\ \text{s.t.} \quad & \sum_{i=1}^{2} y_{it} \leq K \quad \forall\, t = 1, ..., 3 \\ & x_{i,t-1} + y_{it} - D_{it} = x_{it} \quad \forall\, i = 1, 2, t = 1, ..., 3 \\ & x_{i,0} = 0 \quad \forall\, i = 1, 2 \\ & x_{it}, y_{it} \geq 0 \quad \forall\, i = 1, 2, t = 1, ..., 3. \end{aligned}$$

The objective function minimizes the total cost. The first constraint is the capacity constraint. The second constraint is the inventory balancing constraint. The third constraint sets the initial inventory to 0. The last constraint is the nonnegativity constraint.