Operations Research, Spring 2013
Homework 02

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. (15 points) Let S_1 and S_2 be two convex sets.
   (a) (10 points) Prove that $S_1 \cap S_2$, the intersection of S_1 and S_2, is also a convex set.
   Note. In general, if S_1, S_2, ..., S_n are all convex sets, then $\cap_{i=1}^n S_i$ is also a convex set for any $n \in \mathbb{N}$. In fact, this still holds even if n approaches infinity.
   (b) (5 points) Provide an example that $S_1 \cup S_2$, the union of S_1 and S_2, is not a convex set.

2. (15 points) Consider the following mathematical program

$$\begin{array}{ll} \min & (x_1 - 3)^2 + (x_2 - 1)^2 \\ \text{s.t.} & x_1 - x_2 \leq 0 \\ & 2x_1 + x_2 \geq 0. \end{array}$$

   (a) (6 points) Graphically illustrate the feasible region and two isocost lines. Note that the isocost lines of this program are not straight lines; they are actually "curves".
   (b) (6 points) Find the optimal solution of this mathematical program.
   Note. You have not been taught how to solve a nonlinear program. Therefore, design your own way to solve this program.
   (c) (3 points) Is there any extreme point optimal solution? If no, why we need not to have any extreme point optimal solution?

3. (10 points; 5 points each) Consider the following mathematical program

$$\begin{array}{ll} \max & \sin(x) \\ \text{s.t.} & x \geq 0 \\ & x \leq 3\pi. \end{array}$$

   (a) Find all the local minima. Among these local minima, which is (are) the global minimum?
   (b) Find all the local maxima. Among these local maxima, which is (are) the global maximum?

4. (Modified from Problem 3.2.3; 10 points, 5 points each) Leary Chemical manufactures three chemicals: A, B, and C. These chemicals are produced via two production processes: 1 and 2. Running process 1 for an hour costs \$4 and yields 3 units of A, 1 of B, and 1 of C. Running process 2 for an hour costs \$1 and produces 1 unit of A and 1 of B. To meet customer demands, at least 10 units of A, 5 of B, and 3 of C must be produced daily.
   (a) Formulate an LP to determine a daily production plan that minimizes the cost of meeting Leary Chemicals daily demands.
   (b) Graphically solve the LP.

5. (Modified from Problem 3.3.2; 10 points) Consider the following LP:

$$\begin{array}{ll} \max & 4x_1 + x_2 \\ \text{s.t.} & 8x_1 + 2x_2 \leq 16 \\ & 5x_1 + 2x_2 \leq 12 \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

(a) (8 points) Graphically determine whether it has an unique optimal solution, has multiple optimal solutions, is infeasible, or is unbounded.

(b) (1 point) Find the set of binding constraints at the point $(x_1, x_2) = (2, 0)$.

(c) (1 point) Find those binding constraints, if any, at the point $(x_1, x_2) = (1, 3)$.

6. (Modified from Problems 3.3.5 and 3.3.6; 10 points; 5 points each) Answer the following questions and briefly explain your answers.

   (a) True or false: If an LP is unbounded, its feasible region must be unbounded.

   (b) True or false: If an LP has an unbounded feasible region, it must be unbounded.

7. (Modified from Problem 3.3.8; 10 points) Graphically solve the following LP:

$$\begin{aligned} \min \quad & x_1 - x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 6 \\ & x_1 - x_2 \geq 0 \\ & -x_1 + x_2 \geq 3 \\ & x_1, x_2 \geq 0. \end{aligned}$$

8. (Modified from Problem 3.5.5; 10 points) Each day, workers at the Gotham City Police Department work two 6-hour shifts chosen from 12 A.M. to 6 A.M., 6 A.M. to 12 P.M., 12 P.M. to 6 P.M., and 6 P.M. to 12 A.M. The following number of workers are needed during each shift: 12 A.M. to 6 A.M. – 15 workers; 6 A.M. to 12 P.M. – 5 workers; 12 P.M. to 6 P.M. – 12 workers; 6 P.M. to 12 A.M. – 6 workers. Workers whose two shifts are consecutive are paid \$12 per hour; workers whose shifts are not consecutive are paid \$18 per hour. Formulate an LP that can be used to minimize the cost of meeting the daily workforce demands of the Gotham City Police Department.

9. (Modified from Problem 3.Review.11; 10 points) Sunco Oil has refineries in Los Angeles and Chicago. Currently, the Los Angeles refinery can refine up to 2 million barrels of oil per year, and the Chicago refinery up to 3 million. Once refined, oil is shipped to two distribution points: Houston and New York City. Sunco estimates that each distribution point can sell up to 5 million barrels per year. Because of differences in shipping and refining costs, the profit earned (in dollars) per million barrels of oil shipped depends on where the oil was refined and on the point of distribution (see the table below). At this moment, Sunco is considering expanding the capacity of each refinery. Each million barrels of annual refining capacity that is added will cost \$120,000 for the Los Angeles refinery and \$150,000 for the Chicago refinery. Capacity can only be added now but can be used in the future ten years. Formulate a linear program that maximizes Sunco's profits less expansion costs over a ten-year period.

From	To Houston	To New York
Los Angeles	\$20,000	\$15,000
Chicago	\$18,000	\$17,000