Operations Research, Spring 2013
Homework 08

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1. (Modified from Problem 11.1.4; 10 points) Find all the first- and second-order partial derivatives for the function
$$f(x_1, x_2) = x_1^2 e^{x_2}.$$
Then write down its gradient.

2. (Modified from Problem 11.2.1; 10 points) Q & H Company advertises on soap operas and football games. Each second of soap opera ad costs \$50 and each second of football game ad costs \$100. Giving all figures in millions of viewers, if S seconds of soap opera ads are bought, $5\sqrt{S}$ men and $20\sqrt{S}$ women will see the ads. If F seconds of football ads are bought, they will be seen by $17\sqrt{F}$ men and $7\sqrt{F}$ women. Q & H wants at least 40 million men and at least 60 million women to see its ads. Formulate a nonlinear program that can solve Q & H's problem.

3. (Modified from Problem 11.2.9; 10 points) Widgetco produces widgets at plant 1 and plant 2. It costs $20x^{\frac{1}{2}}$ to produce x units at plant 1 and $40x^{\frac{1}{3}}$ to produce x units at plant 2. Each plant can produce as many as 70 units. Each unit produced can be sold for \$10. At most 120 widgets can be sold. Formulate a nonlinear program whose solution will tell Widgetco how to maximize profit.

4. (Modified from Problem 11.2.10; 10 points) Three cities are located at the vertices of an equilateral triangle whose each edge is of length a. An airport is to be built at a location that minimizes the total distance from the airport to the three cities. Formulate a nonlinear program whose solution will tell us where to build the airport.

   Note. To formulate your program, you need to first rewrite the above program description into a more precise one by yourself.

5. (10 points) For the nonlinear program you formulate in Problem 2, is a local optimum always a global optimum? If there is an optimal solution, will there always be an extreme point optimal solution? Please prove your conclusions.

6. (10 points) For the nonlinear program you formulate in Problem 3, is a local optimum always a global optimum? If there is an optimal solution, will there always be an extreme point optimal solution? Please prove your conclusions.

7. (Modified from Problem 11.3.1; 15 points, 3 points each) On the given set S, use the derivatives to determine whether each function is convex, concave, or neither.

   (a) $f(x) = x^3$, $S = [0, \infty)$.
   (b) $f(x) = \frac{1}{x}$, $S = (0, \infty)$.
   (c) $f(x) = \sin(2x)$, $S = [\pi, 2\pi]$.
   (d) $f(x) = x^a$ for some $a \in (0, 1)$, $S = [0, \infty)$.
   (e) $f(x) = 2^x$, $S = \mathbb{R}$.

8. (Modified from Problem 11.4.2; 10 points) If a monopolist produces q units, she can sell all of them at a price $100 - 4q$ dollars per unit. A fixed cost of production 50 will be incurred once any positive amount is produced. The variable per-unit production cost is \$5. What production quantity maximizes the monopolist' profit? If a sales tax of \$2 per unit must be paid by the monopolist, should it increase or decrease production?

9. (15 points, 3 points each) Consider the following primal nonlinear program
$$\begin{aligned} \min \quad & (x_1 - 3)^2 + (x_2 - 1)^2 \\ \text{s.t.} \quad & x_1 - x_2 \leq 0. \end{aligned}$$

(a) Find the primal optimal solution by drawing a graph.

(b) Write down the Lagrangian relaxation of this program. Let λ be the Lagrange multiplier.

(c) Suppose λ is given, what is the optimal solution (as a function of λ) of the Lagrangian?

(d) Let $L(\lambda)$ be the objective value of the optimized Lagrangian relaxation. Show that $L(\lambda)$ is concave in λ.

(e) Formulate and solve the Lagrangian dual, find the optimal λ, and then use the optimal λ to solve the primal.