Operations Research, Spring 2013
Suggested Solution for Homework 11

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1. (a) As $1 + (-1) = (-1) + 1 = 0$, all payoffs in a cell sum to zero. This is thus a zero-sum game.

   (b) Let x_1 and x_2 be player 1's probability of choosing H and T, respectively. Then player 1's problem can be formulated as

$$\begin{aligned} \max \quad & \min\{x_1 - x_2, -x_1 + x_2\} \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & x_1 \geq 0,\ x_2 \geq 0, \end{aligned}$$

   where the minimum functions exists due to the fact that player 2 will try to minimize player 1's expected payoff. To linearize the above nonlinear program, we introduce a new variable u:

$$\begin{aligned} \max \quad & u \\ \text{s.t.} \quad & u \leq x_1 - x_2 \\ & u \leq -x_1 + x_2 \\ & x_1 + x_2 = 1 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

   (c) To solve the linear program we obtained in Part (b), we first utilize the constraint $x_1 + x_2 = 1$ to remove x_2 and obtain an equivalent linear program with only two variables:

$$\begin{aligned} \max \quad & u \\ \text{s.t.} \quad & u \leq 2x_1 - 1 \\ & u \leq 1 - 2x_1 \\ & 0 \leq x \leq 1. \end{aligned}$$

   The graphical approach of the above linear program is depicted in Figure 1. The optimal solution is $(x_1^*, u^*) = (\frac{1}{2}, 0)$ and indeed player 1 should choose each action with probability $\frac{1}{2}$.

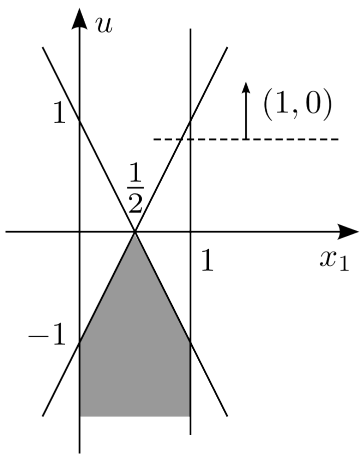

Figure 1: Graphical approach for Problem 1c

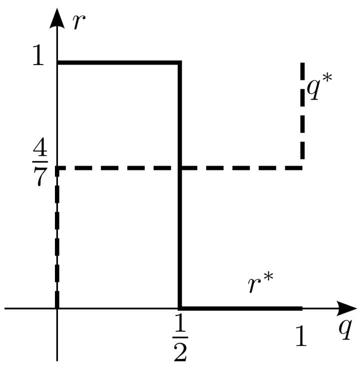

Figure 2: Graphical approach for Problem 2b

2. (a) None of the four pure-strategy action combination is a Nash equilibrium:
   - (H, H) is not because player 2 will deviate ($1 > -1$).

- (H, T) is not because player 1 will deviate ($3 > -1$).
- (T, H) is not because player 1 will deviate ($2 > -1$).
- (T, T) is not because player 2 will deviate ($1 > -1$).

(b) Let q and r be the probabilities for players 1 and 2 to choose H, respectively. Player 1's problem is

$$\begin{aligned} &\max_{0 \leq q \leq 1} \quad 2qr - q(1-r) - (1-q)r + 3(1-q)(1-r) \\ = &\max_{0 \leq q \leq 1} \quad q(7r-4) - 4r + 3, \end{aligned}$$

which results in player 1's best response function $q^*(r) = 0$ if $r < \frac{4}{7}$, 1 if $r > \frac{4}{7}$, and $[0,1]$ if $r = \frac{4}{7}$. This is depicted in Figure 2 as the dashed curve. Following the same procedure, we may find player 2's best response function. However, as player 2's payoffs are identical to those in the original "matching pennies", player 2's best response function must remain the same, which is $r^*(q) = 1$ if $q < \frac{1}{2}$, 0 if $q > \frac{1}{2}$, and $[0,1]$ if $q = \frac{1}{2}$. This is depicted in Figure 2 as the solid curve. The only intersection, which corresponds to the unique Nash equilibrium, is $(q^*, r^*) = (\frac{4}{7}, \frac{1}{2})$.

3. (a) The game matrix is

	firm 1	firm 2
firm 1	$50, 50$	$100, 150$
firm 2	$150, 100$	$75, 75$

(b) Let players 1 and 2 choose firm 1 with probabilities q and r, respectively. Player 1's problem is then formulated as

$$\begin{aligned} &\max_{0 \leq q \leq 1} \quad 50qr + 100q(1-r) + 150(1-q)r + 75(1-q)(1-r) \\ = &\max_{0 \leq q \leq 1} \quad 25q(-5r+1) + 75r + 75. \end{aligned}$$

which results in the best response function

$$q^*(r) = \begin{cases} 1 & \text{if } r < \frac{1}{5} \\ 0 & \text{if } r > \frac{1}{5} \\ [0,1] & \text{if } r = \frac{1}{5} \end{cases}.$$

Similarly, player 2's best response function is

$$r^*(q) = \begin{cases} 1 & \text{if } q < \frac{1}{5} \\ 0 & \text{if } q > \frac{1}{5} \\ [0,1] & \text{if } q = \frac{1}{5} \end{cases}.$$

(c) The two best response functions are depicted in Figure 3, where the solid curve is player 1's best response and the dashed curve is player 2's best response. The three Nash equilibria lie at the intersections $(0, 1)$, $(\frac{1}{5}, \frac{1}{5})$, and $(1, 0)$. In other words, the equilibrium outcome may be (1) player 1 chooses firm 2 and player 2 chooses firm 1, (2) player 1 chooses firm 1 and player 2 chooses firm 2, and (3) both players choose firm 1 with probability $\frac{1}{5}$ and firm 2 with probability $\frac{4}{5}$.

4. (a) The payoff matrix is

	rock	scissor
rock	$0, 0$	$1, -1$
scissor	$-1, 1$	$0, 0$
paper	$1, -1$	$-1, 1$

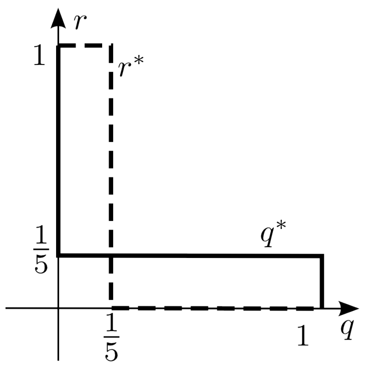

Figure 3: Graphical approach for Problem 3c

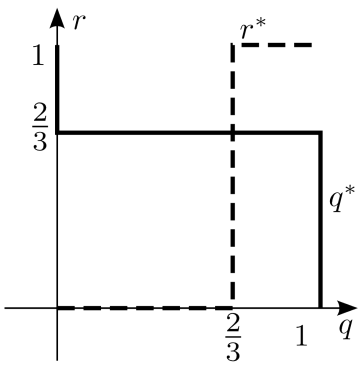

Figure 4: Graphical approach for Problem 4d

(b) For player 1, the action "scissor" is strictly dominated by "rock". To see this, note that when player 2 chooses "rock", player 1 should not choose "scissor" because choosing "rock" is strictly better ($0 > -1$). Moreover, when player 2 chooses "scissor", player 1 should not choose "scissor" because choosing "rock" is still strictly better ($1 > 0$). Therefore, we may remove the action "scissor" from player 1's action space and obtain the reduced game matrix

	rock	scissor
rock	$0, 0$	$1, -1$
paper	$1, -1$	$-1, 1$

(c) Let q and r be the probabilities for players 1 and 2 to choose rock, respectively. Player 1's problem is then formulated as

$$\begin{aligned} &\max_{0 \leq q \leq 1} \quad q(1-r) + (1-q)r - (1-q)(1-r) \\ = &\max_{0 \leq q \leq 1} \quad q(-3r+2) + 2r - 1, \end{aligned}$$

which results in the best response function

$$q^*(r) = \begin{cases} 1 & \text{if } r < \frac{2}{3} \\ 0 & \text{if } r > \frac{2}{3} \\ [0,1] & \text{if } r = \frac{2}{3} \end{cases}.$$

Similarly, player 2's problem is formulated as

$$\begin{aligned} &\max_{0 \leq r \leq 1} \quad -q(1-r) - (1-q)r + (1-q)(1-r) \\ = &\max_{0 \leq r \leq 1} \quad 3r(3q-2) - 2q + 1, \end{aligned}$$

which results in the best response function

$$r^*(q) = \begin{cases} 1 & \text{if } q < \frac{2}{3} \\ 0 & \text{if } q > \frac{2}{3} \\ [0,1] & \text{if } q = \frac{2}{3} \end{cases}.$$

(d) The two best response functions are depicted in Figure 4, where player 1's is depicted as the solid curve and player 2's as the dashed curve. As there is only one intersection, the unique Nash equilibrium is $(q^*, r^*) = (\frac{2}{3}, \frac{2}{3})$ and both players will play rock with probability $\frac{2}{3}$.

(e) Let x be player 1's probability of choosing rock. By recognizing that this is a zero-sum game, player 1's problem can be formulated as

$$\begin{aligned} \max \quad & u \\ \text{s.t.} \quad & u \leq 1 - x \\ & u \leq 2x - 1 \\ & 0 \leq x \leq 1 \end{aligned}$$

and solved by the graphical approach illustrated in Figure 5. The optimal solution (x^*, u^*) tells us that in equilibrium player 1 will indeed choose rock with probability $\frac{2}{3}$.

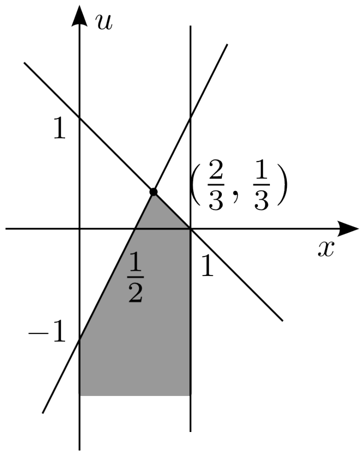

Figure 5: Graphical approach for Problem 4e

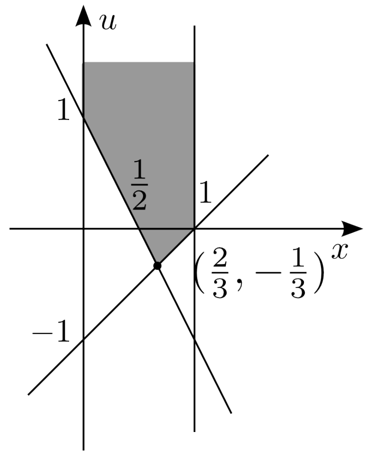

Figure 6: Graphical approach for Problem 4f

(f) Let y be player 2's probability of choosing rock. By recognizing that this is a zero-sum game, player 2's problem can be formulated as

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & v \leq y - 1 \\ & v \leq 1 - 2y \\ & 0 \leq y \leq 1. \end{aligned}$$

and solved by the graphical approach illustrated in Figure 6. The optimal solution (y^*, v^*) tells us that in equilibrium player 2 will indeed choose rock with probability $\frac{2}{3}$.

(g) As we know, a Nash equilibrium of a zero-sum game must correspond to a pair of primal and dual optimal solutions to the two players' optimization problems. As the two players' problems have only one optimal solution, there is only one Nash equilibrium. As that Nash equilibrium is a mixed-strategy one, there is no pure-strategy Nash equilibrium.

5. (a) In equilibrium, both player confess. Their equilibrium payoffs are both −6.

(b) In equilibrium, player 1 chooses Bach and player 2 chooses Bach. Their equilibrium payoffs are 2 and 1, respectively.

(c) In equilibrium, player 1 may choose either Head or Tail and player 2 will choose the opposite. Their equilibrium payoffs are −1 and 1, respectively.

(d) In prisoners' dilemma, being a leader or not does not matter; in BoS, being a leader is beneficial; in matching pennies, being a leader hurts a player. Being a leader is not always beneficial.