IM2010: Operations Research
Linear Programming Formulation
(Chapter 3)

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Introduction

- It is important to learn how to model a practical situation as a linear program.
- This process is typically called **linear programming formulation** or **modeling**.
- We will introduce three types of LP problems, demonstrate how to formulate them, and discuss some important issues.
  - There are certainly many other types of LP problems.
- For large-scale problems, **compact formulations** are used.

Road map

- **Resource allocation**.
- Materials blending.
- Production and inventory.
- Compact formulations.

Resource allocation

- We produce products to sell.
- Each product requires some resources. **Resources are limited**.
- We want to maximize the total sales revenue while ensuring resources are enough.

Resource allocation: the problem

- We may produce desks and tables.
  - Producing a desk requires four units of wood, one hour of labor, and 30 minutes of machine time.
  - Producing a table requires five units of wood, two hours of labor, and 20 minutes of machine time.
- We may sell everything we produce.
- For each day, we have
  - Two workers, each works for eight hours.
  - One machine that can run for eight hours.
  - A supply of 36 units of wood.
- Desks and tables are sold at $800 and $600 per unit, respectively.

Formulation: decision variables

- When we define decision variables, try to answer "what are the **decisions to make**?"
- In this example, the decision we want to make is the production quantities of desks and tables.
- Therefore, we define our decision variables as follows:
- Let

$$x_1 = \text{number of desks produced in a day and}$$
$$x_2 = \text{number of tables produced in a day.}$$

Formulation: objective function

- In the objective function, we write down the quantity that we want to **maximize** or **minimize**.
- In this example, we want to maximize the total sales revenue.
  - Given our decision variables, may we determine the sales revenue?
  - The sales revenue is $800x_1 + 600x_2$.
- The objective function is thus

$$\max \ 800x_1 + 600x_2.$$

Formulation: constraints

- For each **restriction** or **limitation**, we write a constraint.
- Summarizing data into a table typically helps:

Resource	Consumption per		Total supply
	Desk	Table	
Wood	4 units	5 units	36 units
Labor hour	1 hour	2 hours	16 hours
Machine time	30 minutes	20 minutes	8 hours

Formulation: constraints

- The supply of wood is limited:

$$4x_1 + 5x_2 \leq 36.$$

- The number of labor hours is limited:

$$x_1 + 2x_2 \leq 16.$$

- The amount of machine time is limited:

$$30x_1 + 20x_2 \leq 240.$$

  - Use the same unit of measurement!
- Production quantities are **nonnegative**: $x_i \geq 0 \quad \forall i = 1, 2.$

Formulation: the complete formulation

- The complete formulation is

$$
\begin{array}{rrcrcrl}
\max & 800x_1 & + & 600x_2 & & & \\
\text{s.t.} & 4x_1 & + & 5x_2 & \leq & 36 & \text{(wood)} \\
 & x_1 & + & 2x_2 & \leq & 16 & \text{(labor)} \\
 & 30x_1 & + & 20x_2 & \leq & 240 & \text{(machine)} \\
 & \multicolumn{6}{l}{x_i \geq 0 \quad \forall i = 1, 2}
\end{array}
$$

  - **Clearly** define decision variables **in front of** your formulation.
  - Write **comments** after the objective function and constraints.
  - Do not forget nonnegativity constraints.

Formulation: the complete formulation

- We may simplify the formulation:

$$
\begin{array}{rrcrcrl}
\max & 8x_1 & + & 6x_2 & & & \\
\text{s.t.} & 4x_1 & + & 5x_2 & \leq & 36 & \text{(wood)} \\
 & x_1 & + & 2x_2 & \leq & 16 & \text{(labor)} \\
 & 3x_1 & + & 2x_2 & \leq & 24 & \text{(machine)} \\
 & & & & & & x_i \geq 0 \quad \forall i = 1, 2
\end{array}
$$

  - Once we find an optimal solution, please use the **original** objective function in calculating the associated objective value.

Fractional and integer variables

- The optimal solution of this linear program is to produce 6.86 desks or 1.71 tables. Can we?
- Indeed we cannot. Then why linear programming?
  - It always **supports** our decisions. E.g., we may round down to get a feasible solution that is near optimal.
  - In practice, people use mathematical programming typically when the quantities are **large**. Rounding 6.86 may deviate a lot but rounding 68600.86 may be much more acceptable.
  - When it is necessary, we should impose integer constraints on variables and apply **integer programming** (to be covered later in the semester).
- If it is not specified in the problem, using LP is enough.

Road map

- Resource allocation.
- **Materials blending**.
- Production and inventory.
- Compact formulations.

Material blending

- In some situations, we need to determine not only products to produce but also **materials** to input.
- This is because we have some **flexibility** in making the products.
- For example, in making orange juice, we may use orange, sugar, water, etc. Different ways of **blending** these materials results in different qualities of juice.
- The goal is to save money (lower the proportion of expensive materials) while maintaining **quality**.
- This is introduced in Section 3.7 of the textbook.

Material blending: the problem

- We blend materials 1, 2, and 3 to make products 1 and 2.
- The quality of a product, which depends on the proportions of these three materials, must meet the standard:
  - Product 1: at least 40% of material 1; at least 20% of material 2.
  - Product 2: at least 50% of material 1; at most 30% of material 3.
- At most 100 kg of product 1 and 150 kg of product 2 can be sold.
- Prices for products 1 and 2 are $10 and $15 per kg, respectively.
- Costs for materials 1 to 3 are $8, $4, and $3 per kg, respectively.
- Amount of a product made equals the amount of materials input.
- We want to maximize the total profit.

Formulation: decision variables

- Probably our first attempt is to define the following: Let

$$\begin{aligned} x_1 &= \text{kg of product 1 produced,} \\ x_2 &= \text{kg of product 2 produced,} \\ y_1 &= \text{kg of material 1 produced,} \\ y_2 &= \text{kg of material 2 produced, and} \\ y_3 &= \text{kg of material 3 produced.} \end{aligned}$$

- May we express the quality of each product? No!
- We need to specify the amount of material 1 used for product 1, the amount of material 1 used for product 2, etc.
- So we need to **redefine** our decision variables.

Formulation: decision variables

- How about this: Let

$$\begin{aligned}
x_1 &= \text{kg of material 1 used for product 1,} \\
x_2 &= \text{kg of material 1 used for product 2,} \\
x_3 &= \text{kg of material 2 used for product 1,} \\
x_4 &= \text{kg of material 2 used for product 2,} \\
x_5 &= \text{kg of material 3 used for product 1, and} \\
x_6 &= \text{kg of material 3 used for product 2.}
\end{aligned}$$

- The definition is correct and precise, but **not easy to use**.
  - Similar to computer programming: give your variables reasonable names that allow people to know **what they are**.

Formulation: decision variables

- How about this: Let

$$
\begin{aligned}
x_{11} &= \text{kg of material 1 used for product 1,} \\
x_{12} &= \text{kg of material 1 used for product 2,} \\
x_{21} &= \text{kg of material 2 used for product 1,} \\
x_{22} &= \text{kg of material 2 used for product 2,} \\
x_{31} &= \text{kg of material 3 used for product 1, and} \\
x_{32} &= \text{kg of material 3 used for product 2.}
\end{aligned}
$$

- Much better.

Formulation: decision variables

- How to find the production quantities of products and the purchasing quantities of materials?
- Let's summarize the variables into a table:

	Product 1	Product 2
Material 1	x_{11}	x_{12}
Material 2	x_{21}	x_{22}
Material 3	x_{31}	x_{32}

- What are the desired quantities?

Formulation: decision variables

- The desired quantities:

	Product 1	Product 2	Purchasing quantity
Material 1	x_{11}	x_{12}	$x_{11} + x_{12}$
Material 2	x_{21}	x_{22}	$x_{21} + x_{22}$
Material 3	x_{31}	x_{32}	$x_{31} + x_{32}$
Production quantity	$x_{11} + x_{21} + x_{31}$	$x_{12} + x_{22} + x_{32}$	

Formulation: objective function

- Let's write down the total profit.
- Sales revenues depend on the amount of products we sell.
  - How much product 1 may we sell? $x_{11} + x_{21} + x_{31}$.
  - Similarly, we have $x_{12} + x_{22} + x_{32}$ kg of product 2.
- Material costs depend on the amount of materials we purchase.
  - Similarly, we need to buy $x_{11} + x_{12}$ kg of material 1, $x_{21} + x_{22}$ kg of material 2 and $x_{31} + x_{32}$ kg of material 3.
- The objective function is

$$\begin{aligned} &\max\ 10(x_{11} + x_{21} + x_{31}) + 15(x_{12} + x_{22} + x_{32}) \\ &\qquad - 8(x_{11} + x_{12}) - 4(x_{21} + x_{22}) - 3(x_{31} + x_{32}) \\ =\ &\max\ 2x_{11} + 7x_{12} + 6x_{21} + 11x_{22} + 7x_{31} + x_{32}. \end{aligned}$$

Formulation: quality constraints

- In product 1, how to guarantee at least 40% are material 1?

$$\frac{x_{11}}{x_{11}+x_{21}+x_{31}} \geq 0.4.$$

- It is conceptually correct. However, it is **nonlinear**!
- Let's fix the nonlinearity by taking the denominator to the RHS:

$$x_{11} \geq 0.4(x_{11}+x_{21}+x_{31}).$$

  Though equivalent, they are just different.
- We may (but is not required to) choose other format, such as $0.6x_{11} - 0.4x_{21} - 0.4x_{31} \geq 0$ or $3x_{11} - 2x_{21} - 2x_{31} \geq 0$.

Formulation: constraints

- In total we have four quality constraints:
  - $x_{11} \geq 0.4(x_{11} + x_{21} + x_{31})$.
  - $x_{21} \geq 0.2(x_{11} + x_{21} + x_{31})$.
  - $x_{12} \geq 0.5(x_{12} + x_{22} + x_{32})$.
  - $x_{13} \leq 0.3(x_{12} + x_{22} + x_{32})$.

Formulation: constraints

- The demands are limited:

$$x_{11} + x_{21} + x_{31} \leq 100$$

and

$$x_{12} + x_{22} + x_{32} \leq 150.$$

- The quantities are nonnegative:

$$x_{ij} \geq 0 \quad \forall i = 1, ..., 3, j = 1, 2.$$

Formulation: the complete formulation

- The complete formulation is

$$
\begin{aligned}
\max \quad & 10(x_{11} + x_{21} + x_{31}) + 15(x_{12} + x_{22} + x_{32}) \\
& - 8(x_{11} + x_{12}) - 4(x_{21} + x_{22}) - 3(x_{31} + x_{32}) \\
\text{s.t.} \quad & x_{11} \geq 0.4(x_{11} + x_{21} + x_{31}) \\
& x_{21} \geq 0.2(x_{11} + x_{21} + x_{31}) \\
& x_{12} \geq 0.5(x_{12} + x_{22} + x_{32}) \\
& x_{13} \leq 0.3(x_{12} + x_{22} + x_{32}) \\
& x_{11} + x_{21} + x_{31} \leq 100 \\
& x_{12} + x_{22} + x_{32} \leq 150 \\
& x_{ij} \geq 0 \quad \forall i = 1, ..., 3, j = 1, 2.
\end{aligned}
$$

Remarks

- We may need to redefine decision variables when we find they are not enough.
- We may from time to time use multi-dimensional variables.
- We need to remove nonlinear constraints or objective functions, even if we just replace them with equivalent linear ones.

Road map

- Resource allocation.
- Materials blending.
- **Production and inventory**.
- Compact formulations.

Production and inventory

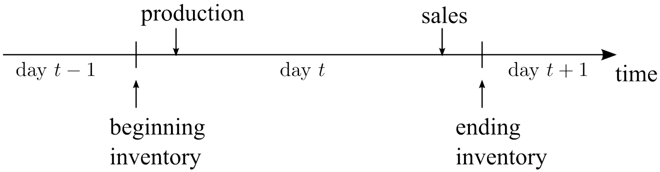
- When we are making decisions, we may need to consider what will happen in the **future**.
- This creates **multi-period** problems.
- In particular, in many cases products produced today may be **stored** and then sold in the future.
  - Maybe production is cheaper today.
  - Maybe the price is higher in the future.
- So the production decision must be jointly considered with the **inventory** decision.
- Introduced in Section 3.10 of the textbook.

Production and inventory: the problem

- Suppose we are going to produce and sell a product in four days.
- For each day, there are different amounts of demands to fulfill.
  - Days 1, 2, 3, and 4: 100, 150, 200, and 170 units, respectively.
- The unit production costs are different for different days:
  - Days 1, 2, 3, and 4: \$9, \$12, \$10, and \$12 per unit, respectively.
- The prices are all **fixed**. So maximizing profits is the same as minimizing costs.

Production and inventory: the problem

- We may store a product and sell it later.
  - The **inventory cost** is \$1 per unit per day.
  - E.g., producing 620 units on day 1 to fulfill all demands costs $9 \times 620 + 1 \times 150 + 2 \times 200 + 3 \times 170 = 6640$ dollars.
- Timing:



  - Beginning inventory + production − sales = ending inventory.
  - Inventory costs are assessed according to **ending inventory**.

Formulation: decision variables

- We need to determine the production quantities: Let

$$x_t = \text{production quantity of day } t, t = 1, ..., 4.$$

- Is that information enough?
  - E.g., given a plan (450, 0, 170, 0), we do not know whether the demand on day 4 is fulfilled with the productions on day 1 or 3.
- So we also need to determine the inventory quantities: Let

$$y_t = \text{ending inventory of day } t, t = 1, ..., 4.$$

  - It is important to specify "ending"!

Formulation: objective function

- We have production costs:

$$9x_1 + 12x_2 + 10x_3 + 12x_4.$$

- We also have inventory costs:

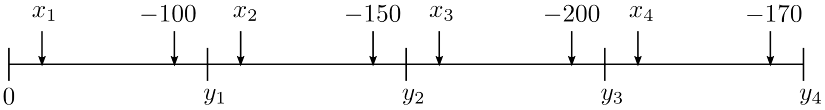
$$1(y_1 + y_2 + y_3 + y_4).$$

- So the objective function is

$$\min \ 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4.$$

Formulation: constraints

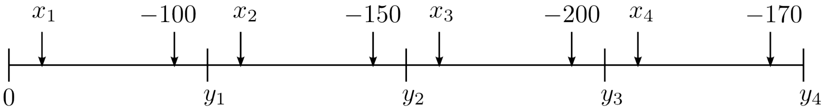
- We need to relate adjacent periods through ending inventories:
  - Day 1: $x_1 - 100 = y_1$.
  - Day 2: $y_1 + x_2 - 150 = y_2$.
  - Day 3: $y_2 + x_3 - 200 = y_3$.
  - Day 4: $y_3 + x_4 - 170 = y_4$.



- This is typically called the **inventory balancing** constraint.

Formulation: constraints

- We must satisfy all the demands at the moment of sales:
  - Day 1: $x_1 \geq 100$.
  - Day 2: $y_1 + x_2 \geq 150$.
  - Day 3: $y_2 + x_3 \geq 200$.
  - Day 4: $y_3 + x_4 \geq 170$.



- Finally, all quantities must be nonnegative.

Formulation: the complete formulation

- The complete formulation is

$$
\begin{aligned}
\min \quad & 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4 \\
\text{s.t.} \quad & x_1 - 100 = y_1 \\
& y_1 + x_2 - 150 = y_2 \\
& y_3 + x_3 - 200 = y_3 \\
& y_3 + x_4 - 170 = y_4 \\
& x_1 \geq 100 \\
& y_1 + x_2 \geq 150 \\
& y_2 + x_3 \geq 200 \\
& y_3 + x_4 \geq 170 \\
& x_t, y_t \geq 0 \quad \forall t = 1, ..., 4.
\end{aligned}
$$

Simplifying the formulation

- Let's look at the demand fulfillment constraints again.
- The first one is $x_1 \geq 100$.
  - But we have the first inventory balancing constraint $x_1 - 100 = y_1$ and the nonnegativity constraint $y_1 \geq 0$. They together imply $x_1 \geq 100$.
  - Similarly, $y_1 + x_2 - 150 = y_2$ and $y_2 \geq 0$ imply $y_1 + x_2 \geq 150$.
- So all demand fulfillment constraints can be removed.

Formulation: the simplified formulation

- The simplified formulation is

$$
\begin{aligned}
\min \quad & 9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4 \\
\text{s.t.} \quad & x_1 - 100 = y_1 \\
& y_1 + x_2 - 150 = y_2 \\
& y_3 + x_3 - 200 = y_3 \\
& y_3 + x_4 - 170 = y_4 \\
& x_t, y_t \geq 0 \quad \forall t = 1, ..., 4.
\end{aligned}
$$

Remarks

- The main idea is to use inventory variables to **connect multiple periods**. Otherwise periods will be unconnected.
- From time to time, we may first write some constraints and then find they are **redundant**.
- There are other ways of formulating this problem. For example, for the production lot on day t, we may split it into those for day t, those for day $t+1$, etc.

Road map

- Resource allocation.
- Materials blending.
- Production and inventory.
- **Compact formulations**.

Compact formulations

- Most problems in practice are of **large scales**.
  - The number of variables and constraints are huge.
- Many variables can be grouped together:
  - E.g., x_t = production quantity of day $t, t = 1, ..., 4$.
- Many constraints can be grouped together:
  - E.g., $x_t \geq 0$ for all $t = 1, ..., 4$.
- In modeling large-scale problems, we must use **compact formulations** to enhance readability and efficiency.

Compact formulations

- In general, we may use the following three instruments:
  - Indices $(i, j, k, ...)$.
  - Summation $(\sum)$.
  - For all $(\forall)$.
- For the joint production-inventory problem, let's write a compact formulation.

Production and inventory

- The problem:
  - We have four periods.
  - In each period, we first produce and then sell.
  - Unsold products become ending inventories.
  - Want to minimize the total cost.
- Indices:
  - Because things will **repeat in each period**, it is natural to use an index for periods. Let $t \in \{1, ..., 4\}$ be the index of periods.
- Now let's make the LP formulation compact.

Compacting the objective function

- The original objective function:
  - min $9x_1 + 12x_2 + 10x_3 + 12x_4 + y_1 + y_2 + y_3 + y_4$.
- We may combine the last four terms:
  - min $9x_1 + 12x_2 + 10x_3 + 12x_4 + \sum_{t=1}^{4} y_t$.
- To combine the first four terms, we may need to create a **parameter set**.
  - Denote $C = [9\ 12\ 10\ 12]$ as the **production cost vector** where C_t is the unit price on day $t, t = 1, ..., 4$.
  - min $\sum_{t=1}^{4} C_t x_t + \sum_{t=1}^{4} y_t$.
  - min $\sum_{t=1}^{4} (C_t x_t + y_t)$.

Compacting the constraints

- The original constraints:
  - $x_1 - 100 = y_1$,
  - $y_1 + x_2 - 150 = y_2$,
  - $y_2 + x_3 - 200 = y_3$, and
  - $y_3 + x_4 - 170 = y_4$.
- Again, let's create a parameter set and group these constraints.
  - Denote $D = [100\ 150\ 200\ 170]$ as the demand vector where D_t is the demand on day $t, t = 1, ..., 4$.
- For day $t, t = 2, ..., 4 : y_{t-1} + x_t - D_t = y_t$.
  - We cannot apply this to day 1 as y_0 is undefined!
  - How may we group the four constraints together?

Compacting the constraints

- Let's define y_0: Let

$$y_t = \text{ending inventory of day } t, t = 0, ..., 4.$$

  - The ending inventory of day 0, by definition, should be the initial inventory of day 1.
- Then we may write

$$y_{t-1} + x_t - D_t = y_t \quad \forall t = 1, ..., 4$$

  as the set of inventory balancing constraints.
  - Certainly we need to set up the initial inventory: $y_0 = 0$.

The complete compact formulation

- The compact formulation is

$$\begin{aligned} \min \quad & \sum_{t=1}^{4} (C_t x_t + y_t) \\ \text{s.t.} \quad & y_{t-1} + x_t - D_t = y_t \quad \forall t = 1, ..., 4 \\ & y_0 = 0 \\ & x_t, y_t \geq 0 \quad \forall t = 1, ..., 4. \end{aligned}$$

  - **Do not forget** "$\forall t = 1, ..., 4$"! Without that, the formulation is just wrong.
  - Nonnegativity constraints for multiple sets of variables can be combined to save some "≥ 0".

Parameters v.s. variables

- We need to define decision variables.
  - **Let** (**Define**) x_t = production quantity on day $t, t = 1, ..., 4$.
- We need to create parameter sets.
  - **Denote** $C = [9\ 12\ 10\ 12]$ **as** the production cost vector where C_t is the unit production cost on day $t, t = 1, ..., 4$.
- For parameters, we just **define their names**. We do not **define parameters**. They exist before we give them names!
- Variables do not exist before we define them.
- One convention is to:
  - Use **lowercase** letters for variables (e.g., x_t).
  - Use **uppercase** letters for parameters (e.g., C_t).

Parameters v.s. variables

- When creating parameter sets, it is fine to write only:

  Denote $C = [9\ 12\ 10\ 12]$ as the production cost vector.

  - C_t is naturally its t^{th} element and has no ambiguity.
  - The **values** should be indicated when defining the name.
- It is also fine to write

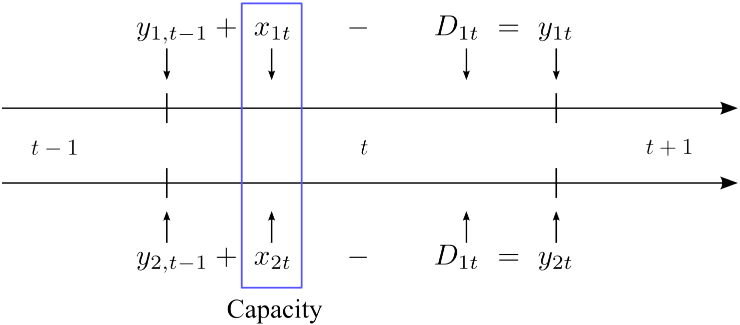
  Denote C_t as the unit production cost on day $t, t = 1, ..., 4$.

  - Do not need to specify values.
  - Need to specify **range** through **indices**.
- In either case, we should indicate the **physical meaning**.

Another production-inventory example

- Suppose we will produce and sell N products in T periods.
- We are given
  - The unit production cost of each product in each period,
  - The demand of each product in each period,
  - The holding cost of each product per period,
  - The machine time for producing one unit of each product, and
  - The capacity (measured in total machine time) of each day.
- How to write an LP that can minimize the total cost?

Another production-inventory example



Another production-inventory example

- Let $N = \{1, 2, ..., 10\}$, $T = \{1, 2, ..., 100\}$, and $T_0 = T \cup \{0\}$.
- For variable, let

$$x_{it} = \text{ production quantity of product } i \text{ in period } t, i \in N, t \in T, \text{ and}$$
$$y_{it} = \text{ ending of product } i \text{ in period } t, i \in N, t \in T_0.$$

- For parameters, denote

C_{it} as the unit production cost of product i in period $t, i \in N, t \in T$,
H_i as the unit inventory cost per period $i \in N$,
D_{it} as the demand of product i in period $t, i \in N, t \in T$,
P_i as the machine time required for product $i, i \in N$, and
K_t as the machine time capacity in period $t, t \in T$.

Another production-inventory example

- The problem can then be formulated as

$$\begin{aligned}
\min \quad & \sum_{i \in N} \sum_{t \in T} C_{it} X_{it} + \sum_{i \in N} H_i \sum_{t \in T} y_{it} \\
\text{s.t.} \quad & y_{i,t-1} + x_{i,t-1} - D_{i,t-1} = y_{it} \quad \forall i \in N, t \in T \\
& y_{i0} = 0 \quad \forall i \in N \\
& \sum_{i \in N} P_{it} x_{it} \leq K_t \quad \forall t \in T \\
& x_{it}, y_{it} \geq 0 \quad \forall i \in N, t \in T.
\end{aligned}$$