IM2010: Operations Research
Inventory Models
(Chapters 15 and 16)

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Road map

- **Introduction**.
- The EOQ model.
- Variants of the EOQ model.
- The newsvendor model.

What are inventory?

- For almost all firms producing or purchasing products to sell, they need **inventory**.
- Why inventory?
  - If each batch of production or procurement requires some **fixed costs**, we will increase the batch size to save money.
  - If **demand is uncertain**, inventory provides a buffer for supply-demand mismatch.
- Key questions in the manufacturing and retailing industries regarding inventory include:
  - When to do replenishment?
  - How much to replenish?
  - From which suppliers?
- In this session, we introduce fundamental OR models that make the optimal inventory decisions.

An LP-based inventory model

- We have seen the following inventory model:
  - We have T periods with different demands.
  - In each period, we first produce and then sell.
  - Unsold products become ending inventories.
  - We want to minimize the total cost.
  - In period t, C_t is the unit production cost, D_t is the unit production quantity, and H is the unit holding cost per period.
- The formulation is

$$\begin{aligned} \min \quad & \sum_{t=1}^{T} (C_t x_t + H y_t) \\ \text{s.t.} \quad & y_{t-1} + x_t - D_t = y_t \quad \forall t = 1, ..., T \\ & y_0 = 0 \\ & x_t, y_t \geq 0 \quad \forall t = 1, ..., T. \end{aligned}$$

Categories of inventory models

- The previous model is an example of a **periodic review system** with **deterministic demands**.
  - Replenishment can occur at most once per "period".
  - All future demands are perfectly predicted.
- In a **continuous review system**, one may replenish at any time point.
- When we are facing inventory decisions, in general there are four types of inventory systems:

Demand	Review time	
	Periodic	Continuous
Deterministic	1	2
Random	3	4

Two NLP-based inventory models

- We will introduce two NLP-based inventory models:
  - The **economic order quantity** (EOQ) model.
  - The **newsvendor** model.
- They are basic, fundamental, widely applied.
- They are the foundations of most advanced inventory models.
- They are the applications of (single-variate) NLP.
- How to categorize them?

Demand	Review time	
	Periodic	Continuous
Deterministic	The LP-based model	EOQ
Random	Newsvendor	N/A

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Motivating example

- IM Airline uses 500 taillights per year. It purchases these taillights from a manufacturer at a unit price $500.
- Taillights are consumed at a **constant rate** throughout a year.
- Whenever IM Airline places an order, an **ordering cost** of $5 is incurred regardless of the order quantity.
- The **holding cost** is 2 cents per taillight per month.
- IM Airline wants to minimize the total cost, which is the sum of ordering, purchasing, and holding costs.
- How much to order? When to order?
  - What is the benefit of having a small or large order?

The EOQ model

- IM Airline's question may be answered with the economic order quantity (EOQ) model.
- In this model, we try to find the order quantity that is the most economic.
  - In particular, we want to find a **balance** between the ordering cost and holding cost.
- Technically, we will formulate an NLP whose optimal solution is the optimal order quantity.

Assumptions of the EOQ model

- Demand is deterministic and occurs at a constant rate.
- Regardless the order quantity, a fixed ordering cost is incurred.
- No shortage is allowed.
- The ordering lead time is zero.
- The inventory holding cost is constant.

Parameters and the decision variable

- Parameters:

$$\begin{aligned} D &= \text{annual demand (units)}, \\ K &= \text{unit ordering cost (\$)}, \\ h &= \text{unit holding cost per year (\$), and} \\ p &= \text{unit purchasing cost (\$)}. \end{aligned}$$

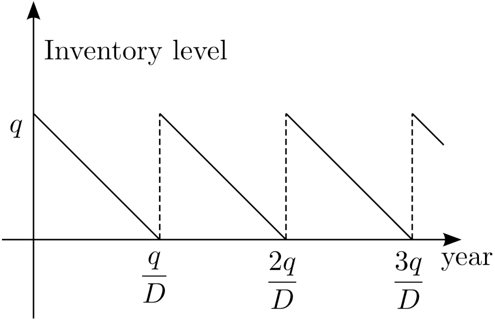
- Decision variable:

$$q = \text{ order quantity per order (units)}.$$

- Objective: Minimizing annual total cost.
- For all our calculations, we will use **one year** as our time unit. Therefore, D can be treated as the demand **rate**.

Inventory level by time

- To formulate the problem, we need to understand how the **inventory level** is affected by our decision.
- Based on the EOQ assumptions, we will always place an order when the inventory level is zero.
- As inventory is consumed at a constant rate, the inventory level vary in time in the following way:



Annual costs

- Annual holding cost $= h(\frac{q}{2}) = \frac{hq}{2}$.
  - For a time period, the holding cost is h times the area under the curve over that time period.
  - For one year, the length of the time period is 1 and the inventory level is $\frac{q}{2}$ **in average**.
- Annual purchasing cost $= pD$.
  - We need to buy D units regardless the order quantity.
- Annual ordering cost $= K(\frac{D}{q}) = \frac{KD}{q}$.
  - The number of orders in a year is $\frac{D}{q}$.
- Annual total cost $= TC(q) = \frac{KD}{q} + pD + \frac{hq}{2}$.

Nonlinear optimization

- The NLP for optimizing the ordering decision is

$$\min_{q \geq 0} \; TC(q) = \frac{KD}{q} + pD + \frac{hq}{2}.$$

- We have

$$TC'(q) = -\frac{KD}{q^2} + \frac{h}{2}, \text{ and}$$
$$TC''(q) = \frac{2KD}{q^3} > 0.$$

  Therefore, $TC(q)$ is convex in q.

Optimizing the order quantity

- Let q^* be the quantity satisfying the FOC:

$$TC'(q^*) = -\frac{KD}{(q^*)^2} + \frac{h}{2} = 0 \quad \Rightarrow \quad q^* = \sqrt{\frac{2KD}{h}}.$$

- As this quantity is feasible, it is optimal.
- The resulting annual holding and ordering cost is $\sqrt{2KDh}$.
- The optimal order quantity q^* is the EOQ. It is:
  - Increasing in the ordering cost K.
  - Increasing in the annual demand D.
  - Decreasing in the holding cost h.
  - Unaffected by the purchasing cost p.

  Why?

Example

- IM Airline uses 500 taillights per year.
- The ordering cost is $5 per order.
- The holding cost is 2 cents per unit per month.
- Taillights are consumed at a constant rate.
- No shortage is allowed.
- Questions:
  - What is the EOQ?
  - How many orders to place in each year?
  - What is the order cycle time (time between two orders)?

Example: the optimal solution

- The EOQ is

$$q^* = \sqrt{\frac{2(5)(500)}{(0.24)}} \approx \sqrt{20833.33} \approx 144.34 \text{ unit.}$$

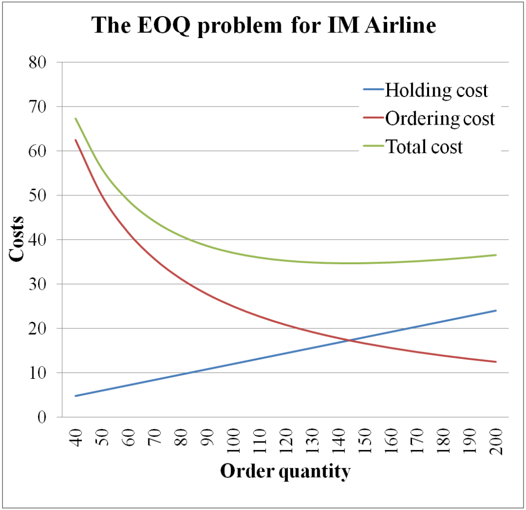
  - Make sure that time units are consistent!
- The average number of orders in a year is $\frac{500}{q^*} \approx 3.464$ orders.
- The order cycle time is

$$T^* = \frac{1}{3.464} \approx 0.289 \text{ year} \approx 3.464 \text{ months.}$$

- The number of orders in a year and the order cycle time are the same! Is it a coincidence?

Example: cost analysis

- The annual holding cost is $\frac{hq^*}{2} \approx \$17.32$.
- The annual ordering cost is $\frac{KD}{q^*} \approx \$17.32$.
- The two costs are identical! Is it a coincidence?



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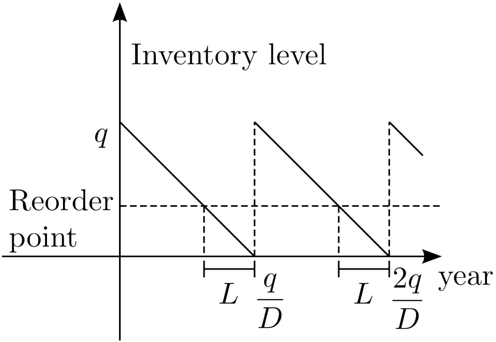
Nonzero lead time

- What if there is an **ordering lead time** $L > 0$?
  - This means after we place an order, we will receive the product after L year.
- In this case, we want to calculate the **reorder point**, which is the inventory level at which an order should be placed.



Reorder points

- When to order?
- Let R be the reorder point. We want to calculate R such that we receive products exactly when we have **no inventory**.
- If $L \leq T^*$:

$$R = LD.$$

  - $T*$ is the order cycle time.
  - L must be measured in years!
- If $L \geq T^*$:

$$R = D(L - kT^*)$$

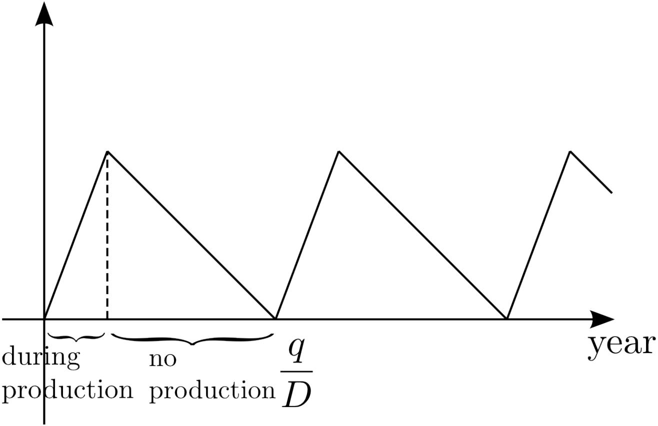
for some $k \in \mathbb{N}$ such that $0 \leq L - kT^* \leq T^*$.

Economic production quantity (EPQ)

- When products are produced rather than purchased, typical they are "received" at a continuous rate.
- The model that finds the optimal **production lot size** is called the **economic production quantity** (EPQ) model.
- Under the assumption that the product is **produced at a constant rate** of r units per year, what lot size minimizes the total cost?

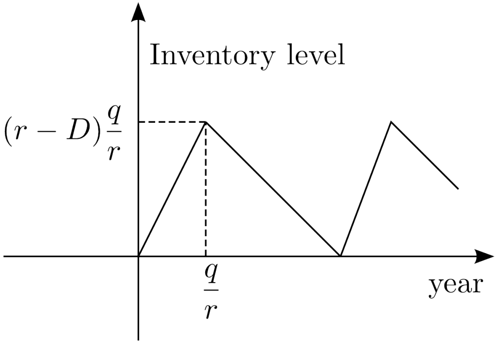
Economic production quantity (EPQ)

- The inventory level now looks like:



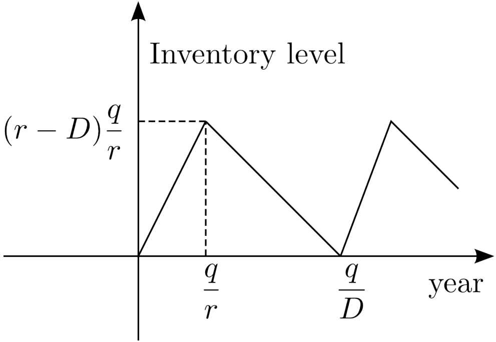
Economic production quantity

- Suppose we choose q as our production lot size.
- In the production time, the demand increases at the rate $r - D$.
  - While we produce at the rate r, we also consume at the rate D.
- The length of the production time is $\frac{q}{r}$ year. Why?
- So the maximum inventory level (achieved at the end of a production period) is $(r - D)\frac{q}{r}$.



Economic production quantity

- Still, the amount we produce in a lot will be depleted in $\frac{q}{D}$ year.



Economic production quantity

- The annual holding cost now becomes

$$h\left[\frac{q(r-D)}{2r}\right].$$

  - The average inventory level is $\frac{1}{2}[(r-D)\frac{q}{r}]$.
- The annual setup cost is still $K(\frac{D}{q})$.
- The purchasing cost still does not affect the decision.
- The total holding and setup cost is:

$$\frac{hq(r-D)}{2r}+\frac{KD}{q}.$$

Economic production quantity

- Note that this is the same as the EOQ model

$$\frac{hq}{2} + \frac{KD}{q}.$$

if we let $h(\frac{r-D}{r}) = h(1 - \frac{D}{r})$ be the **effective holding cost**.

- The optimal production lot size is thus

$$q^* = \sqrt{\frac{2KD}{h\left(1 - \frac{D}{r}\right)}},$$

which is the EPQ we desire.

Example

- IM Auto needs to produce 10000 cars per year.
- Each car requires \$2000 to produce.
- Each run requires \$200 to set up.
- Annual **holding cost rate** is 25%:
  - The holding cost per car per year is $\frac{\$2000}{4} = \$500$.
- The production rate is 25000 cars per year.
- What is the EPQ and optimal cycle time?

Example

- The EPQ is
$$\sqrt{\frac{2(200)(10000)}{500(1-\frac{10000}{25000})}} = 115.47 \text{ cars.}$$
- The optimal cycle time is
$$\frac{1}{\frac{10000}{115.47}} \approx 0.012 \text{ year } \approx 4.21 \text{ days.}$$
- Will the annual holding cost and annual setup cost still be identical? Why?

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Newsvendor model

- In some situations, people sell **perishable products**.
  - Perishable products will become valueless after the **selling season** is end.
  - E.g., newspapers become valueless after one day.
  - High-tech goods are valueless once the next generation is offered.
  - Fashion goods become valueless when they become out of fashion.
- For perishable products, sometimes the seller only have **one chance** for replenishment.
  - E.g., a small newspaper seller can order only once and obtain those newspapers only at the morning of each day.
- Often sellers of perishable products face **uncertain demands**.
- The question is **how many** products one should prepare for the entire selling season.

Newsvendor model

- Let D be the uncertain demand.
- Let F and f be the cdf and pdf of D (assuming D is continuous).
- Let c_o be the **overage cost** and c_u be the **underage cost**.
  - They are also called overstocking and understocking costs.
  - They are the costs for prepare too many or too few products.
- We want to find an order quantity that minimize the expected total overage and underage costs.
  - As demands are uncertain, we try to minimize the **expectation** of the random cost.

Components of overage and underage costs

- Components of overage and underage costs may include:
  - Sales revenue r for each unit sold.
  - Purchasing cost c for each unit purchased.
  - Salvage value a for each unit unsold.
  - Disposal fee p for each unit unsold.
  - Shortage cost s for each unit of shortage.
- With these quantities, we have
  - The overage cost $c_o = c + p - a$.
  - The underage cost $c_u = r - c + s$.

Formulation of the newsvendor problem

- Let q be the order quantity (inventory level).
- Let d be the realization of demand.
  - D is a random variable and d is a realized value of D.
- Then the cost is

$$c(q,d) = \begin{cases} c_o(q-d) & \text{if } q \geq d \\ c_u(d-q) & \text{if } q < d. \end{cases}$$

- Now, the expected total cost is

$$c(q,D) = \mathbb{E}\Big[c_o(q-d)^+ + c_u(d-q)^+\Big],$$

where $x^+ = \max(x,0)$.

Convexity of the cost function

- We want to find a quantity q that solves

$$\min_{q\geq 0} \ \mathbb{E}\Big[c_o(q-d)^+ + c_u(d-q)^+\Big].$$

- By assuming that D is continuous, the cost function $c(q, D)$ is

$$\int_0^q \big[c_o(q-x) + c_u \cdot 0\big] f(x)dx + \int_q^\infty \big[c_o \cdot 0 + c_u(x-q)\big] f(x)dx$$

$$= c_o \int_0^q (q-x) f(x)dx + c_u \int_q^\infty (x-q) f(x)dx$$

$$= c_o\left[q\int_0^q f(x)dx - \int_0^q x f(x)dx\right] + c_u\left[\int_q^\infty x f(x)dx - q\int_q^\infty f(x)dx\right]$$

$$= c_o\left[qF(q) - \int_0^q x f(x)dx\right] + c_u\left[\int_q^\infty x f(x)dx - q\big(1-F(q)\big)\right].$$

Convexity of the cost function

- The first-order derivative of $c(q, D)$ is

$$
\begin{aligned}
&c'(q, D) \\
&= c_o\Big[F(q) + qf(q) - qf(q)\Big] + c_u\Big[- qf(q) - (1 - F(q)) + qf(q)\Big] \\
&= c_o\big[F(q)\big] - c_u\big[1 - F(q)\big].
\end{aligned}
$$

- The second-order derivative of $c(q, D)$ is

$$c''(q, D) = c_o f(q) - c_u f(q) = f(q)(c_u + c_0) > 0.$$

- So $c(q, D)$ is convex in q.

Optimizing the order quantity

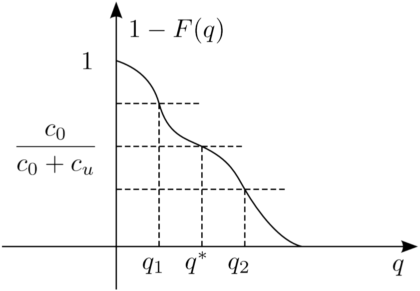
- Let q^* be the order quantity that satisfies the FOC, we have

$$c_o F(q^*) - c_u(1 - F(q^*)) = 0$$
$$\Rightarrow F(q^*) = \frac{c_u}{c_o + c_u} \quad \text{or} \quad 1 - F(q^*) = \frac{c_o}{c_o + c_u}.$$

- Such q^* must be positive (for regular demand distributions). So q^* is optimal.
- Note that to minimize the expected total cost, the seller should **intentionally** create some shortage!
  - The optimal probability of having a shortage is $\frac{c_o}{c_o+c_u}$.

Determinants of the optimal quantity

- The probability of having a shortage, $1 - F(q)$, is decreasing in q.
- The optimal quantity q^* is:
  - Decreasing in c_o: When c_o increases, the optimal quantity moves from q^* to q_1.
  - Increasing in c_u: When c_u increases, the optimal quantity moves from q^* to q_2.



Example 1

- Suppose for a newspaper:
  - The unit purchasing cost is \$5.
  - The unit retail price is \$15.
  - The demand is uniformed distributed between 20 to 50.
- Overage cost $c_o = 5$ and underage cost $c_u = 15 - 5 = 10$.
- The optimal order quantity q^* satisfies

$$1 - F(q^*) = \left(1 - \frac{q^* - 20}{50 - 20}\right) = \frac{5}{5 + 10} \quad \Rightarrow \quad \frac{50 - q^*}{30} = \frac{1}{3},$$

  which implies $q^* = 40$.
- If the unit purchasing cost increases to \$6, we need $\frac{50-q^{**}}{30} = \frac{6}{15}$ and thus $q^{**} = 36$.
  - As the purchasing cost increases, we **dislike overstocking** more. Therefore, we stock less.

Example 2

- Suppose for one kind of apple:
  - The unit purchasing cost is \$15, the unit retail price is \$21, and the unit salvage value is \$1.
  - The demand $D \sim \text{ND}(90, 20)$, i.e., D is normally distributed with mean 90 and standard deviation 20.
  - Overage cost $c_o = 15 - 1 = 14$ and underage cost $c_u = 21 - 15 = 6$.
- The optimal order quantity q^* satisfies

$$\Pr(D < q^*) = \frac{6}{14 + 6} \quad \Rightarrow \quad \Pr\left(Z < \frac{q^* - 90}{20}\right) = 0.3,$$

  where $Z \sim \text{ND}(0, 1)$.
- By looking at a standard normal probability table or using any Statistical software, we find $\Pr(Z < -0.5244) = 0.3$, which implies $\frac{q^* - 90}{20} = -0.5244$ and thus $q^* = 79.512$.
  - As the purchasing cost is so high, we want to **reject more than half** of the consumers!