IM2010: Operations Research
Game Theory: Static Games (Part 1)
(Chapter 14 and Gibbons (1992))

Ling-Chieh Kung

Department of Information Management
National Taiwan University

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The beginning of game theory

- So far we have focused on decision making problems with only one decision maker.
- **Game theory** provides a rigorous framework for analyzing **multi-player** decision making problems.
- While it has been implicitly discussed in Economics for more than 200 years, game theory is established as a field in 1934.
  - In 1934, John von Neumann and Oskar Morgenstern published a book *Theory of games and economic behaviors*.
- Since then, game theory has been widely studied, applied, and discussed in mathematics, economics, operations research, industrial engineering, and computer science.
  - Actually almost all fields of social sciences and business have game theory involved in.
  - The Nobel Prizes in economic sciences have been honored to game theorists in 1994, 1996, 2001, 2007, and 2012.

Road map

- **Introduction**.
- Nash equilibrium.
- Retailer competitions.

Prisoners' dilemma: story

- A and B broke into a grocery store and stole some money. Before police officers caught them, they hided those money carefully without leaving any evidence. However, a monitor got their images when they broke the window.
- They were kept in two separated rooms. Each of them were offered two choices: **Denial or confession**.
  - If both of them deny the fact of stealing money, they will both get one month in prison.
  - If one of them confesses while the other one denies, the former will be set free while the latter will get nine months in prison.
  - If both confesses, they will both get six months in prison.
- They **cannot communicate** and they must make their choices **simultaneously**.
- What will they do?

Prisoners' dilemma: matrix representation

- We may use the following matrix to summarize this "game":

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

  - There are two **players**, player 1 chooses actions in rows and player 2 chooses actions in columns.
  - For each combination of actions, the two numbers are the **payoffs** of the two players under their actions: the first for player 1 and the second for player 2.
  - E.g., if both prisoners deny, they will both get one month in prison, which is represented by a payoff of -1.
  - E.g., if prisoner 1 denies and prisoner 2 confesses, prisoner 1 will get 0 month in prison (and thus a payoff 0) and prisoner 2 will get 9 months in prison (and thus a payoff -9).

Prisoners' dilemma: solution

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

- Prisoner 1 thinks:
  - "If he denies, I should confess."
  - "If he confesses, I should still confess."
  - "I see! I should confess anyway!"
- For prisoner 2, the situation is the same and he will also **confess**.
- The **solution** of this game, i.e., the **outcome**, is that both prisoner will confess.
  - This is people's **prediction** of this game.
- This outcome can be "improved" if they can **cooperate**.

Prisoners' dilemma: discussions

- A game like the prisoners' dilemma in which all players choose their actions **simultaneously** is called a **static game**.
- This question (with a different story) was first formally raised by Professor Tucker (one of the names in the KKT condition) in a seminar.
- In this game, confession is said to be a **dominant strategy**.
- It illustrates that **lack of coordination** can result in a **lose-lose** outcome.
  - This situation is termed as **socially inefficient**.
- Interestingly, even if they promised each other to deny once they are caught, this promise is **non-credible**. Both of them will still confess to maximize their payoffs.

Prisoners' dilemma: Advertising game

- Two companies are competing in a market.
- At this moment, they both earn four million dollars per year.
- Each of them may choose to advertise with a cost of three million per year:
  - If one advertises while the other does not, she earns nine millions and the competitor earns one million.
  - If both advertise, both will earn six millions.

	Advertise	Be silent
Advertise	$3, 3$	$6, 1$
Be silent	$1, 6$	$4, 4$

- What will they do?

Prisoners' dilemma: Arms race

- Two countries are neighbors.
- Each of them may choose to develop a new weapon:
  - If one does so while the other one keep the current status, the former's payoff is 20 and the latter's payoff is -100.
  - If both do this, however, their payoffs are both -10.

	MW	CS
MW	$-10, -10$	$20, -100$
CS	$-100, 20$	$0, 0$

- What will they do?

Predicting the outcome of other games

- How about games that are not the prisoners' dilemma? Do we have a systematic way to predict the outcome?
- What will be the outcome (a combination of actions chosen by the two players) of the following game?

	Left	Middle	Right
Up	1, 0	1, 2	0, 1
Down	0, 3	0, 1	2, 0

Eliminating strictly dominated options

- We may apply the same trick we used to solve the prisoners' dilemma.
- For player 2, playing Middle **dominates** playing Right. So we may **eliminate** the column of Right without eliminating any possible outcome:

	Left	Middle	Right
Up	$1, 0$	$1, 2$	$0, 1$
Down	$0, 3$	$0, 1$	$2, 0$

$\rightarrow$

	Left	Middle
Up	$1, 0$	$1, 2$
Down	$0, 3$	$0, 1$

Eliminating strictly dominated options

- Now, player 1 knows that player 2 will never play Right.
- Facing the reduced game, player 1 finds that playing Down is dominated by playing Up.
- The row of Down can thus be eliminated:

	Left	Middle
Up	$1, 0$	$1, 2$
Down	$0, 3$	$0, 1$

$\rightarrow$

	Left	Middle
Up	$1, 0$	$1, 2$

- Knowing that player 1 will only choose Up, player 2 will simply choose Middle.
- The outcome of this game will be that player 1 chooses Up and player 2 chooses Middle.

Eliminating strictly dominated options

- In game theory, options are typically called **strategies**.
- The above idea is called **iterative elimination** of **strictly dominated strategies**.
- It solves some games. However, is also fails to solve some others.
- Consider the following game "Matching pennies":

	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

- What may we do when no more strategies can be eliminated?
- In 1950, John Nash formalized the concept of **equilibrium solutions**, which are called **Nash equilibria** nowadays.[1]

[1] He did that as a Ph.D. students, when he was 21 years old.

Road map

- Introduction.
- **Nash equilibrium**.
- Retailer competitions.

Nash equilibrium: definition

- The most fundamental equilibrium concept, Nash equilibrium, is defined as follows:

Definition 1

For an n-player game, let S_i be player i's action space and u_i be player i's utility function, $i = 1, ..., n$. An action profile $(s_1^*, ..., s_n^*)$, $s_i^* \in S_i$, is a Nash equilibrium if

$$u_i(s_1^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*) \\ \geq u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*)$$

for all $s_i \in S_i$, $i = 1, ..., n$.

- In other words, s_i^* solves

$$\max_{s_i \in S_i} \quad u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*).$$

Nash equilibrium: an example

- Consider the following game in which no strategy/action is strictly dominated:

	L	C	R
T	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
B	3, 5	3, 5	6, 6

- What is a Nash equilibrium?
  - (T, L) is not: Player 1 will deviate to M or B.
  - (T, C) is not: Player 2 will deviate to L or R.
  - (B, R) is: No one will unilaterally deviate.
  - Any other Nash equilibrium?

Nash equilibrium as a solution concept

	L	C	R
T	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
B	3, 5	3, 5	6, 6

- In a static game, a Nash equilibrium is a reasonable outcome.
  - Imagine that the players play this game **repeatedly**.
  - If they happen to be in a Nash equilibrium, no one has the incentive to **unilaterally deviate**, i.e., to change her action while all others keep their actions.
  - If they do not, at least one will deviate. This process will continue until a Nash equilibrium is reached.
- For example, if they starts at (T, L), eventually they will stop at (B, R), the unique Nash equilibrium of this game.

Nash equilibrium: More examples

- Is there any Nash equilibrium of the prisoners' dilemma?

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

- Is there any Nash equilibrium of the game "BoS"?
  - Battle of sexes.
  - Bach or Stravinsky.

	Bach	Stravinsky
Bach	$2, 1$	$0, 0$
Stravinsky	$0, 0$	$1, 2$

- Is there any Nash equilibrium of the matching pennies game?

	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

Road map

- Introduction.
- Nash equilibrium.
- **Retailer competitions**.

Cournot Competition

- In 1838, Antoine Cournot introduced the following **quantity competition** between two retailers.
- Let q_i be the production quantity of firm i, $i = 1, 2$.
- Let $P(Q) = a - Q$ be the market-clearing price for an aggregate demand $Q = q_1 + q_2$.
- Unit production cost of both firms is $c < a$.
- Our questions are:
  - In this environment, what will these two firms do?
  - Is the outcome satisfactory?
  - What is the difference between duopoly and monopoly (or equivalently, decentralization or integration).

Cournot Competition

- Players: 1 and 2.
- Action spaces: $S_i = [0, \infty)$ for $i = 1, 2$.
- Utility functions:

$$u_i(q_1, q_2) = q_i[a - (q_i + q_{3-i}) - c], i = 1, 2.$$

- As for an outcome, we look for a Nash equilibrium.
- If (q_1^*, q_2^*) is a Nash equilibrium, it must satisfy

$$\max_{q_1 \in [0,\infty)} u_1(q_1, q_2^*) = q_1[a - (q_1 + q_2^*) - c] \text{ and}$$
$$\max_{q_2 \in [0,\infty)} u_2(q_1^*, q_2) = q_2[a - (q_1^* + q_2) - c].$$

Solving the Cournot competition

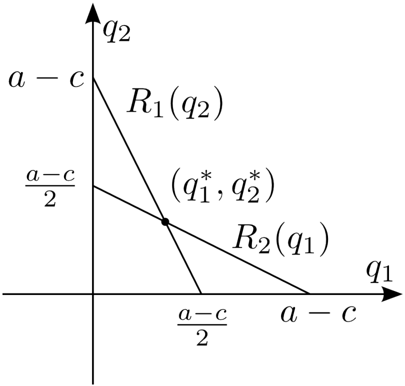
- For firm 1's problem, we first see that it is a convex program:
  - $u_1'(q_1, q_2^*) = a - q_1 - q_2^* - c - q_1$.
  - $u_2''(q_1, q_2^*) = -2 < 0$.
- The FOC condition suggests $q_1^* = \frac{1}{2}(a - q_2^* - c)$. As long as $q_2^* < a - c$, q_1^* is optimal for firm 1.
- Similarly, $q_2^* = \frac{1}{2}(a - q_1^* - c)$ is firm 2's optimal decision as long as $q_1^* < a - c$.
- So if (q_1^*, q_2^*) is a Nash equilibrium, it must satisfy

$$q_1^* = \frac{1}{2}(a - q_2^* - c) \quad \text{and} \quad q_2^* = \frac{1}{2}(a - q_1^* - c).$$

- The unique solution to this system is $q_1^* = q_2^* = \frac{a-c}{3}$.
  - Does this solution make sense?
  - This is indeed the unique Nash equilibrium as $\frac{a-c}{3} < a - c$.

Best responses

- Another way of solving this game is to use the **best response** functions.
  - Given the other player's any decision, what is my optimal decision?
- Firm 1's best response to firm 2 is $R_1(q_2) = \frac{1}{2}(a - q_2 - c)$.
- Similarly, firm 2's best response is $R_2(q_1) = \frac{1}{2}(a - q_1 - c)$.
- A Nash equilibrium always lies on an **intersection** of the two best response functions.



Distortion due to decentralization

- Suppose the two firms' are **integrated** together to jointly choose the aggregate production quantity.
- They together solve

$$\max_{Q \in [0,\infty)} \quad Q[a - Q - c],$$

whose optimal solution is $Q^* = \frac{a-c}{2}$.

- Note that $Q^* = \frac{a-c}{2} < \frac{2(a-c)}{3} = q_1^* + q_2^*$.
- Why does a firm intend to **increase** its production quantity under decentralization?

Inefficiency due to decentralization

- May these firms improve their profitability with integration?
- Under decentralization, firm i earns

$$\pi_i^D = \frac{(a-c)}{3}\left[a - \frac{2(a-c)}{3} - c\right] = \left(\frac{a-c}{3}\right)\left(\frac{a-c}{3}\right) = \frac{(a-c)^2}{9}.$$

- Under integration, the two firms earn

$$\pi^C = \frac{(a-c)}{2}\left[a - \frac{a-c}{2} - c\right] = \left(\frac{a-c}{2}\right)\left(\frac{a-c}{2}\right) = \frac{(a-c)^2}{4}.$$

- $\pi^C > \pi_1^D + \pi_2^D$: The integrated system is more **efficient**.
- Through appropriate profit splitting, both firm earns more.
  - Integration is a **win-win** solution!

Inefficiency due to decentralization

- How about consumers?
- Under decentralization, the aggregate quantity is $\frac{2(a-c)}{3}$ and the market-clearing price is $\frac{a-c}{3}$.
- Under integration, the aggregate quantity is $\frac{a-c}{2}$ and the market-clearing price is $\frac{a-c}{2}$.
- Under decentralization, **more** consumers buy this product with a **lower** price.
- Consumers **benefits from competition**.
- Integration benefits the firms but hurts consumers.

The two firms' prisoners' dilemma

- Now we know it is the two firms' best interests to together produce $Q = \frac{a-c}{2}$.
- What if we suggest each of them to choose $q_1' = q_2' = \frac{a-c}{4}$?
- This results in $Q = \frac{a-c}{2}$, which maximizes the total profit.
- However, this is **not** a Nash equilibrium:
  - "If the other firm chooses $q' = \frac{a-c}{4}$, I will move to

$$q'' = R(q') = \frac{1}{2}(a - q' - c) = \frac{3(a-c)}{8}.$$

  - So both firms will have incentives to unilaterally deviate.
- These two firms are engaged in a prisoners' dilemma!

Bertrand competition

- In 1883, Joseph Bertrand considered another format of retailer competition: They choose **prices** instead of quantities.
- Firm i chooses price p_i, $i = 1, 2$.
- Firm i's demand quantity is

$$q_i = a - p_i + bp_{3-i}, i = 1, 2.$$

  - $b \in [0, 1)$ measures the **intensity of competition** is: The larger b, the more intense the competition.
  - Why $b < 1$?
- Unit production cost $c < a$.

Solving the Bertrand competition

- Suppose (p_1^*, p_2^*) is a Nash equilibrium.
- For firm 1, p_1^* must be an optimal solution of

$$\max_{p_1 \in [0,\infty)} \pi_1\big(p_1, p_2^*\big) = \big(a - p_1 + bp_2^*\big)(p_1 - c).$$

  It can be verified that $p_1^* = \frac{1}{2}(a + bp_2^* + c)$.
- Similarly, $p_2^* = \frac{1}{2}(a + bp_1^* + c)$.
- The unique Nash equilibrium is $p_1^* = p_2^* = \frac{a+c}{2-b}$.
  - Does this solution make sense?

Distortion due to decentralization

- Under integration, the two firms together choose a **single price** P to solve

$$\max_{P \in [0,\infty)} 2(a - P + bP)(P - c),$$

whose optimal solution P^* satisfies the FOC

$$\begin{aligned}&(-1 + b)(P^* - c) + a - P^* + bP^* = 0\\ \Leftrightarrow\ &(-1 + b)P^* + a + c(1 - b) = 0\\ \Leftrightarrow\ &P^* = \frac{a + c(1 - b)}{2(1 - b)}.\end{aligned}$$

- Is $P^* > p_1^* = p_2^*$?

$$P^* > p_1^* \ \Leftrightarrow \frac{a + c(1 - b)}{2(1 - b)} > \frac{a + c}{2 - b} \Leftrightarrow a > c(1 - b).$$

Is $a > c(1 - b)$ always true?