# IM2010: Operations Research
# Game Theory: Static Games (Part 2)
# (Chapter 14 and Gibbons (1992))

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# Road map

- **Mixed strategies**.
- Zero-sum games.
- Zero-sum games and LP duality.

# Mixed strategy

- Choosing a single action deterministically is said to implement a **pure strategy**.
- A **mixed strategy** for player $i$ is a **probability distribution** over the strategy space $S_i$.
  - She **randomizes** her choice of actions with the distribution.
  - E.g., in the matching penny game, player 1's mixed strategy is a probability distribution $(q, 1-q)$, where $\Pr(\text{Head}) = q$ and $\Pr(\text{Tail}) = 1-q$.
- Formally, if all the strategy spaces are finite and of size $K_i$:

**Definition 1**

*A mixed strategy for player $i$ is a vector $p_i = (p_{i1}, ..., p_{iK_i})$, where $0 \leq p_{ij} \leq 1$ for all $j = 1, ..., K_i$ and $\sum_{j=1}^{K_i} p_{ij} = 1$.*

# Mixed-strategy Nash equilibrium

- A profile is a **mixed-strategy Nash equilibrium** if no player can unilaterally deviate (modify her own mixed strategy) and obtain a strictly higher **expected** utility.
- Let's use the matching penny game as an example.

| | Head | Tail |
|---|---|---|
| Head | $1, -1$ | $-1, 1$ |
| Tail | $-1, 1$ | $1, -1$ |

  - Let $(q, 1-q)$ be player 1's mixed strategy.
  - Let $(r, 1-r)$ be player 2's mixed strategy.

# Mixed-strategy Nash equilibrium

- Under their strategies, player 1 will earn:
  - $u_1(H, H) = 1$ with probability $qr$.
  - $u_1(H, T) = -1$ with probability $q(1-r)$.
  - $u_1(T, H) = -1$ with probability $(1-q)r$.
  - $u_1(T, T) = 1$ with probability $(1-q)(1-r)$.
- Player 1's expected utility is

$$\begin{aligned} & v_1(q, r) = \mathbb{E}[u_1(q, r)] \\ = & \; qru_1(H, H) + q(1-r)u_1(H, T) \\ & + (1-q)ru_1(T, H) + (1-q)(1-r)u_1(T, T) \\ = & \; qr + (1-q)(1-r) - q(1-r) - (1-q)r \\ = & \; 4qr - 2q - 2r + 1 = 2q(2r-1) - 2r + 1. \end{aligned}$$

- Similarly, player 2's expected utility is

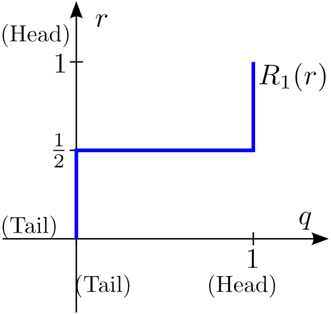
$$v_2(q, r) = -4qr + 2q + 2r - 1 = 2r(-2q+1) + 2q - 1.$$

# Mixed-strategy Nash equilibrium

- For player 1, let $q^* = R_1(r)$ be the best response that maximizes

$$v_1(q, r) = 2q(2r - 1) - 2r + 1.$$

  - If $r < \frac{1}{2}$, $R_1(r) = 0$.
  - If $r > \frac{1}{2}$, $R_1(r) = 1$.
  - If $r = \frac{1}{2}$, $R_1(r) = [0, 1]$ ($q$ does not matter).

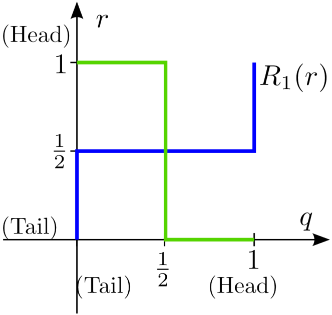


# Mixed-strategy Nash equilibrium

- For player 2, the best response that maximizes

$$v_2(q,r) = -4qr + 2q + 2r - 1 = 2r(-2q+1) + 2q - 1.$$

is $r^* = R_2(q) = 1$ if $q < \frac{1}{2}$, 0 if $q > \frac{1}{2}$, and $[1, 0]$ if $q = \frac{1}{2}$.




- The unique mixed-strategy Nash equilibrium is $(q^*, r^*) = (\frac{1}{2}, \frac{1}{2})$.

# BoS

- Consider the game BoS as another example.

| | Bach | Stravinsky |
|---|---|---|
| Bach | $2, 1$ | $0, 0$ |
| Stravinsky | $0, 0$ | $1, 2$ |

- There are two pure-strategy Nash equilibria. Which two?
  - They are also mixed-strategy Nash equilibria.
  - Is there other mixed-strategy Nash equilibrium?
- Mixed strategies:
  - Let $(q, 1-q)$ be player 1's mixed strategy: $\Pr(B) = q = 1 - \Pr(S)$.
  - Let $(r, 1-r)$ be player 2's mixed strategy: $\Pr(B) = r = 1 - \Pr(S)$.

# BoS

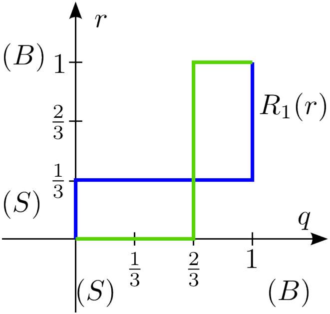
| | Bach | Stravinsky |
|---|---|---|
| Bach | $2, 1$ | $0, 0$ |
| Stravinsky | $0, 0$ | $1, 2$ |

- Player 1's expected utility is $q(3r-1)+1-r$.
- Player 2's expected utility is $r(3q-2)+2(1-q)$.
- The best response functions are

$$R_1(r) = \begin{cases} 0 & \text{if } r < \frac{1}{3} \\ 1 & \text{if } r > \frac{1}{3} \\ [1,0] & \text{if } r = \frac{1}{3} \end{cases} \text{ and } R_2(q) = \begin{cases} 0 & \text{if } r < \frac{2}{3} \\ 1 & \text{if } r > \frac{2}{3} \\ [1,0] & \text{if } r = \frac{2}{3} \end{cases}.$$

# BoS

- The two best response curves have three intersections!




- So there are three mixed-strategy Nash equilibria:
  - $(q^*, r^*) = (0, 0)$, $(\frac{2}{3}, \frac{1}{3})$, and $(1, 1)$.
  - Two of them are pure-strategy Nash equilibria: $(0, 0)$ means both choosing $S$ and $(1, 1)$ means both choosing $B$.

# Mixed strategies over more actions

- Consider the game "Rock, paper, scissor":

| | R | P | S |
|---|---|---|---|
| R | $0, 0$ | $-1, 1$ | $1, -1$ |
| P | $1, -1$ | $0, 0$ | $-1, 1$ |
| S | $-1, 1$ | $1, -1$ | $0, 0$ |

- When a player has three actions, a mixed strategy is described with two variables.
  - E.g., player 1's mixed strategy is $(q_1, q_2, 1 - q_1 - q_2)$.
- When a player's action space is infinite (e.g., those players in the Cournot competition), a mixed strategy is a continuous probability distribution.

# Existence of (mixed-strategy) Nash equilibrium

- In his work in 1950, John Nash proved the following theorem regarding the **existence** of Nash equilibrium:

**Proposition 1**

*For a static game, if the number of players is finite and the action spaces are all finite, there exists at least one mixed-strategy Nash equilibrium.*

  - This is a sufficient condition. Is it necessary?

# Road map

- Mixed strategies.
- **Zero-sum games**.
- Zero-sum games and LP duality.

# Zero-sum games

- For some games, one's **success** is the other one's **failure**.
  - When one earns \$1, the other one loses \$1.
- These games are called **zero-sum games**.
  - The sum of all players' payoffs are always zero under any action profile is zero.
- What is the optimal strategy in a zero-sum game?
  - One's optimal strategy is to **destroy** the other one.

# Zero-sum games

- As an example, the following game is a zero-sum game:

| | L | C | R |
|---|---|---|---|
| T | 4, −4 | 4, −4 | 10, −10 |
| M | 2, −2 | 3, −3 | 1, −1 |
| B | 6, −6 | 5, −5 | 7, −7 |

- For a zero-sum game, we typically remove player 2's payoff:

| | L | C | R |
|---|---|---|---|
| T | 4 | 4 | 10 |
| M | 2 | 3 | 1 |
| B | 6 | 5 | 7 |

  - Player 1 wants to maximize her payoff.
  - Player 2 wants to minimize player 1's payoff.

# Player 1's problem

- How to solve a zero-sum game?
  - The idea of Nash equilibrium still applies. However, the unique structure of zero-sum games allows us to solve them differently.
- Player 1 thinks:
  - If I choose T, he will choose L or C. I get 4.
  - If I choose M, he will choose R. I get 1.
  - If I choose B, he will choose C. I get 5.
- For each of player 1's actions, what he may get in equilibrium can only be the **row minimum**.

| | L | C | R | Row min |
|---|---|---|---|---|
| T | 4 | 4 | 10 | 4 |
| M | 2 | 3 | 1 | 1 |
| B | 6 | 5 | 7 | 5 |

# Player 2's problem

- Player 2 thinks:
  - If I choose L, she will choose B. She get 6.
  - If I choose C, she will choose B. She get 5.
  - If I choose R, she will choose T. She get 10.
- For each of player 2's actions, what player 1 may get in equilibrium must be the **column maximum**.

| | L | C | R | Row min |
|---|---|---|---|---|
| T | 4 | 4 | 10 | 4 |
| M | 2 | 3 | 1 | 1 |
| B | 6 | 5 | 7 | 5 |
| Column max | 6 | 5 | 10 | |

- In equilibrium, player 1 **maximizes the row minimum** and player 2 minimizes the column maximum.
- The unique Nash equilibrium is (B, C).

# Saddle points

- For a zero-sum game, a pure-strategy Nash equilibrium is called a **saddle point**.
- While there may not exist a pure-strategy Nash equilibrium for a general game, this also holds for a zero-sum game.
  - Any example?
- Is there any condition for a pure-strategy Nash equilibrium to exist in a zero-sum game?

# Existence of saddle points

| | L | C | R | R. min |
|---|---|---|---|---|
| T | 4 | 4 | 10 | 4 |
| M | 2 | 3 | 1 | 2 |
| B | 6 | 5 | 7 | 5 |
| C. max | 6 | 5 | 10 | |

| | H | T | R. min |
|---|---|---|---|
| H | 1 | −1 | −1 |
| T | −1 | 1 | −1 |
| C. max | 1 | 1 | |

- For the previous example with a pure-strategy Nash equilibrium,

$$\max\{\text{row minima}\} = 5 = \min\{\text{column maxima}\}.$$

- For the zero-sum game matching penny with no pure-strategy Nash equilibrium,

$$\max\{\text{row minima}\} = 1 \neq -1 = \min\{\text{column maxima}\}.$$

# Existence of saddle points

- Is there any relationship between the existence of saddle points and the values of max{row minima} and min{column maxima}?

**Proposition 2**

*For a two-player zero-sum game, if*

$$\max\{\textit{row minima}\} = \min\{\textit{column maxima}\},$$

*an intersection of a* max{*row minima*} *and a* min{*column maxima*} *is a saddle point.*

- To prove this, we rely on linear programming. In particular, we will apply **strong duality**.

# Road map

- Mixed strategies.
- Zero-sum games.
- **Zero-sum games and LP duality**.

# Mixed strategies for zero-sum games

- For a zero-sum game:
  - A pure-strategy Nash equilibrium (i.e., saddle point) may not exist.
  - A mixed-strategy Nash equilibrium must exist.
- How do players choose their mixed strategies?
- Recall that when a saddle point exists:
  - Player 1 chooses a row to maximize row minimum.
  - Player 2 chooses a column to minimize the column maximum.
- In general:
  - Player 1 chooses a row to maximize the **expectation** of row payoffs **under player 2's mixed strategy**.
  - Player 2 chooses a column to minimize the expectation of column payoffs under player 1's mixed strategy.

# Mixed strategies for zero-sum games

- Suppose player 1's mixed strategy is $x = (x_1, x_2, x_3)$:

| | L | C | R |
|---|---|---|---|
| T (with probability $x_1$) | 4 | 4 | 10 |
| M (with probability $x_2$) | 2 | 3 | 1 |
| B (with probability $x_3$) | 6 | 5 | 7 |
| Expected column payoff | $4x_1 + 2x_2 + 6x_3$ | $4x_1 + 3x_2 + 5x_3$ | $10x_1 + x_2 + 7x_3$ |

- Player 2 will find the smallest expected column maximum.
- Therefore, Player 1 should solve

$$\begin{aligned} \max \quad & \min\{4x_1 + 2x_2 + 6x_3, 4x_1 + 3x_2 + 5x_3, 10x_1 + x_2 + 7x_3\} \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0 \quad \forall i = 1, ..., 3. \end{aligned}$$

# Linearization of player 1's problem

$$\begin{aligned} \max \quad & \min\{4x_1 + 2x_2 + 6x_3, 4x_1 + 3x_2 + 5x_3, 10x_1 + x_2 + 7x_3\} \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0 \quad \forall i = 1, ..., 3. \end{aligned}$$

- Player 1's problem is nonlinear.
- However, it is equivalent to the following linear program:

$$\begin{aligned} \max \quad & v \\ \text{s.t.} \quad & v \leq 4x_1 + 2x_2 + 6x_3 \\ & v \leq 4x_1 + 3x_2 + 5x_3 \\ & v \leq 10x_1 + x_2 + 7x_3 \\ & x_1 + x_2 + x_3 = 1 \\ & x_i \geq 0 \quad \forall i = 1, ..., 3. \end{aligned}$$

# Player 2's problem

- Suppose player 2's mixed strategy is $y = (y_1, y_2, y_3)$.
- Following the same logic, player 2 solves the linear program

$$\begin{aligned} \min \quad & u \\ \text{s.t.} \quad & u \geq 4y_1 + 4y_2 + 10y_3 \\ & u \geq 2y_1 + 3y_2 + y_3 \\ & u \geq 6y_1 + 5y_2 + 7y_3 \\ & y_1 + y_2 + y_3 = 1 \\ & y_i \geq 0 \quad \forall i = 1, ..., 3. \end{aligned}$$

# Duality between the two players

- The two players' problems can be rewritten to

$$
\begin{array}{rrcrcrcrcc}
z^* = \max & & & & & & & v & & \\
\text{s.t.} & -4x_1 & - & 2x_2 & - & 6x_3 & + & v & \leq & 0 \\
 & -4x_1 & - & 3x_2 & - & 5x_3 & + & v & \leq & 0 \\
 & -10x_1 & - & x_2 & - & 7x_3 & + & v & \leq & 0 \\
 & x_1 & + & x_2 & + & x_3 & & & = & 1 \\
 & \multicolumn{9}{l}{x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ v \text{ urs.}}
\end{array}
$$

$$
\begin{array}{rrcrcrcrcc}
w^* = \min & & & & & & & u & & \\
\text{s.t.} & -4y_1 & - & 4y_2 & - & 10y_3 & + & u & \geq & 0 \\
 & -2y_1 & - & 3y_2 & - & y_3 & + & u & \geq & 0 \\
 & -6y_1 & - & 5y_2 & - & 7y_3 & + & u & \geq & 0 \\
 & y_1 & + & y_2 & + & y_3 & & & = & 1 \\
 & \multicolumn{9}{l}{y_1 \geq 0,\ y_2 \geq 0,\ y_3 \geq 0,\ u \text{ urs.}}
\end{array}
$$

- This is a **primal-dual pair**!

# Duality between the two players

- For a two-player zero-sum game, if an LP finds player 1's optimal strategy, its **dual** finds player 2's optimal strategy.
  - A pair of primal and dual optimal solutions $x^*$ and $y^*$ form a mixed-strategy Nash equilibrium.
- Some examples in business:
  - Two competing retailers sharing a fixed amount of consumers.
  - A retailer and a manufacturer negotiating the price of a product.
- Can any of these two LPs be infeasible or unbounded?
  - No! Because a mixed-strategy Nash equilibrium **always exists**.
  - So these two LPs must both have optimal solutions.

# Existence of saddle points

- Now we are ready to prove the theorem regarding the existence of saddle points:

  *For a two-player zero-sum game, if*

  $$\max\{\textit{row minima}\} = \min\{\textit{column maxima}\},$$

  *an intersection of a* $\max\{\textit{row minima}\}$ *and a* $\min\{\textit{column maxima}\}$ *is a saddle point.*

# Existence of saddle points

- First of all, note that choosing a single row or column corresponds to a feasible primal or dual solution:
  - Choosing a single row is for player 1 to implement a pure strategy (by setting the corresponding $x_i = 1$ and $x_k = 0$ for all $k \neq i$).
  - This is a feasible solution to the primal LP.
  - Similarly, choosing a single column corresponds to a feasible solution to the dual LP with $y_j = 1$ and $y_k = 0$ for all $k \neq j$.
- Suppose max{row minima} = min{column maxima} is satisfied:
  - Suppose this occurs at row $i$ and column $j$.
  - Let $x^*$ be the primal solution with $x_i^* = 1$ and $x_k^* = 0$ for all $k \neq i$.
  - Let $y^*$ be the dual solution with $y_j^* = 1$ and $y_k^* = 0$ for all $k \neq j$.
  - As the condition is satisfied, $z^* = w^*$ for two feasible solutions. By strong duality, these two feasible solutions are both optimal.
- A pair of primal-dual optimal solutions form a mixed-strategy Nash equilibrium. As $x_i^* = y_j^* = 1$, $x^*$ and $y^*$ form a saddle point.