Operations Research, Fall 2014
Midterm Exam

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Name: ____________________ Student ID: ____________________

Note. You do not need to return these problem sheets. Write down all your answers on the answer sheets provided to you.

1. (15 points) Consider the linear program

$$
\begin{aligned}
\max \quad & 2x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 4 \\
& x_1 + x_2 = 3 \\
& x_1 \text{ urs.}, x_2 \geq 0.
\end{aligned}
$$

   (a) (5 points) Find the phase-I linear program.

   (b) (10 points) Use the simplex method with the two-phase implementation to solve the linear program. For variable selection, use the smallest index rule.

2. (15 points) An undirected graph $G = (V, E)$ is given with V as the set of nodes and E as the set of undirected edges. We want to select a subset of edges S such that no node is the endpoint of more than one edge in S. More precisely, for each $v \in V$, the number of edges with v as one of the endpoints is at most one. We want to maximize the number of edges selected in S.

   (a) (5 points) On a complete graph, there is an edge between any pair of two nodes. For a complete graph having n nodes, what is the maximum number of edges that we may select?

   (b) (10 points) Given a general undirected graph $G = (V, E)$, formulate a linear integer program that maximizes the number of selected edges. This problem has nothing to do with Part (a).

3. (20 points) A manager is to assign five jobs to five workers. A job cannot be split and must be assigned to a worker. The cost for worker i to do job j is c_{ij}, $i = 1, ..., 5$, $j = 1, ..., 5$. Each worker can be assigned at most two jobs. The manager wants to minimize the total cost.

   (a) (5 points) Formulate the manager's problem as a network flow problem. You may choose any network flow problem introduced in this course.

   (b) (5 points) Write down a linear integer program that describes your formulation in Part (a). Is the coefficient matrix totally unimodular? Explain why.

   (c) (10 points) Suppose jobs can be split but a job can be assigned to at most two workers. When a job is split, it can be split into two pieces with no restriction on the relative sizes. The cost a worker pays to do a job is proportional to the share that she/he is responsible for. Formulate another linear integer program for the new problem.

4. (10 points) A manufacturer orders a raw material from a supplier. Let q be the order quantity (units) of each order. The consumption rate of the material depends on the inventory level I: When $I \geq \frac{q}{2}$, the demand rate is $2D$ (units per year); otherwise, the demand rate is D. The ordering cost is \$$K$ per order and the holding cost is \$$h$ per unit per year. Find the optimal order quantity q^* that minimizes the annual ordering and holding cost.

5. (10 points; 5 points each) Answer the following questions.

   (a) For an EPQ problem with monthly demand 1000 units, production rate 400 per week, holding cost \$1 per unit per month, and setup cost \$50 per lot, what is the EPQ? Assume that each month has four weeks (so each year has 48 weeks).

   (b) For a newsvendor problem with daily demand normally distributed with mean 200 and variance 400, unit purchasing cost \$16, unit sales price \$50, and unit disposal fee \$4, what is the newsvendor quantity?

6. (10 points) A retailer sells products 1 and 2 at supply quantities q_1 and q_2. For product i, the market-clearing price is

$$p_i = a_i - b_i q_i, \quad i = 1, 2,$$

   where $a_i > 0$ and $b_i > 0$ for $i = 1, 2$. The retailer sets q_1 and q_2 to maximize its total profit with the constraint that $q_1 \leq q_2$. For this problem, let's ignore the nonnegativity constraints ($q_1 \geq 0$ for $i = 1, 2$) as they will be nonbinding at an optimal solution.

   (a) (2 points) Formulate the retailer's problem.

   (b) (3 points) Find a necessary and sufficient condition for the constraint to be binding at an optimal solution. Explain why.

   (c) (5 points) Solve the retailer's problem.

7. (10 points) Consider the following nonlinear program

$$\begin{aligned} \max \quad & x_1 \\ \text{s.t.} \quad & x_1 + x_2 \leq 0 \\ & x_1^2 + x_2^2 \geq 18. \end{aligned}$$

   (a) (2 points) Graphically solve the nonlinear program.

   (b) (2 points) Determine whether the nonlinear program is a convex program. Explain why.

   (c) (3 points) Does $(-3, 3)$ satisfy the KKT condition? Explain why.

   (d) (3 points) Suppose the second constraint becomes $x_1^2 + x_2^2 \leq 18$, does $(-3, 3)$ satisfy the KKT condition for the new NLP? Explain why.

8. (10 points) A company will produce and sell products in the following T periods. The demand quantity, unit production cost, unit sales price in period t are D_t, C_t, and P_t, respectively. In each period, the company produces and obtains products before selling products. Therefore, the company may fulfill the demand in a period by products produced in that period. Unsold products may be stored for future sales while unfulfilled demands should be fulfilled by the end of period T. At the end of each period, each unsold product incurs a holding cost H per unit per period. Moreover, each unit of unfulfilled demand incurs a shortage cost S per unit per period. Note that because shortage is allowed, the sales quantity in a period needs not to be identical with the demand quantity. Formulate a linear program that maximizes the company's total profit.