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Suggested Solution for Homework 1

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3. (a) We use Gauss-Jordan elimination to solve the equations:

$$\left[\begin{array}{ccc|c} 0 & 2 & 2 & 4 \\ 1 & 2 & 1 & 4 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 2 & 2 & 4 \\ 0 & 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & -2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

The linear system has a unique solution $(x_1, x_2, x_3) = (1, 1, 1)$.

(b) We use Gauss-Jordan elimination to find the inverse of the matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 4 & 1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -4 & 1 & 0 \\ 0 & 1 & 1 & -3 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -4 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & -4 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right].$$

Therefore, the inverse is

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

(c) To find constants c_1, c_2, and c_3 such that $c_1 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, we use Gauss-Jordan elimination to solve the problem:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 4 & 1 & 2 & 0 \\ 3 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Since $c_1 = c_2 = c_3 = 0$ is the only solution, the column vectors of the matrix are linearly independent. Please note that what we are doing is to find the rank of the matrix. As the matrix has full rank, the three column vectors are linearly independent.

(d) We use Gauss-Jordan elimination to check the number of non-zero row vectors:

$$\begin{bmatrix} 1 & 3 & 0 & 1 \\ 2 & -1 & 2 & 3 \\ 4 & 5 & 2 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -7 & 2 & 1 \\ 0 & -7 & 2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & -7 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there are two non-zero row vectors, the rank of the matrix is 2.

4. (a) $\mathbb{E}[X] = \sum_{x=1}^{6}[x\Pr(X = x)] = \frac{1+2+\cdots+6}{6} = 3.5$.

(b) Let μ be the expectation of X ($\mathbb{E}[X]$), $\text{Var}(X) = \mathbb{E}[(X-\mu)^2] = \sum_{x=1}^{6}[(x-\mu)^2\Pr(X = x)] = \frac{35}{12}$.

(c) $\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = 7$.

(d) $\Pr(X_1 = X_2) = \sum_{i=1}^{6} \Pr(X_1 = X_2 = i) = \frac{6}{36} = \frac{1}{6}$.

5. (a) We reformulate the model to

$$\begin{aligned} \max \quad & 500x_1 + 400x_2 + 200x_3 \\ \text{s.t.} \quad & 4x_1 + 2x_2 + 3x_3 \leq 5 \\ & x_2 + x_3 \leq 1 \\ & x_i \in \{0,1\} \quad \forall i = 1,2,3. \end{aligned}$$

The best solution is $(x_1, x_2, x_3) = (1, 0, 0)$. To maximize the sales revenue, we should only sell the textbook of *Calculus*, and we will earn \$500.

(b) Let

$$x_1 = \begin{cases} 1 & \text{if textbook } \textit{Calculus} \text{ is brought to sell} \\ 0 & \text{otherwise} \end{cases},$$

$$x_2 = \begin{cases} 1 & \text{if textbook } \textit{Computer Programming} \text{ is brought to sell} \\ 0 & \text{otherwise} \end{cases},$$

$$x_3 = \begin{cases} 1 & \text{if textbook } \textit{Operations Research} \text{ is brought to sell} \\ 0 & \text{otherwise} \end{cases}, \text{ and}$$

$$x_4 = \begin{cases} 1 & \text{if textbook } \textit{Optimation} \text{ is brought to sell} \\ 0 & \text{otherwise} \end{cases}.$$

Our model is

$$\begin{aligned} \max \quad & 500x_1 + 400x_2 + 200x_3 + 200x_4 \\ \text{s.t.} \quad & 4x_1 + 2x_2 + 3x_3 + 2x_4 \leq 5 \\ & x_2 + x_3 \leq 1 \\ & x_i \in \{0,1\} \quad \forall i = 1, ..., 4. \end{aligned}$$

The best solution is $(x_1, x_2, x_3, x_4) = (0, 1, 0, 1)$. To maximize the sales revenue, we should sell the textbooks of *ComputerProgramming* and *Optimation*, and we will earn \$600.