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Suggested Solution for Homework 3

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1. The standard form is

$$\begin{aligned} \min \quad & -3x_1 + x_2 - x_3 \\ \text{s.t.} \quad & -x_1 + x_4 = 3 \\ & x_1 - x_2 + x_3 + x_5 = 4 \\ & -2x_1 - x_2 + x_3 = 3 \\ & x_i \geq 0 \quad \forall i = 1, ..., 5. \end{aligned}$$

2. The standard form is

$$\begin{aligned} \max \quad & 3x_1 + 2x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 - x_3 = 100 \\ & x_1 + x_2 + x_4 = 80 \\ & x_1 - x_5 = 40 \\ & x_i \geq 0 \quad \forall i = 1, ..., 5, \end{aligned}$$

where x_3, x_4,and x_5 are slack variables.

(a) Since in the standard form we have five variables and three constraints, there should be three basic variables and two nonbasic variables in a basic solution. The ten possible ways to choose two (nonbasic) variables to be 0 are listed in the table below.

x_1	x_2	x_3	x_4	x_5	Basic feasible solution?
0	0	-100	80	-40	No
0	100	0	-20	-40	No
0	80	-20	0	-40	No
0	N/S	N/S	N/S	0	No
50	0	0	30	10	Yes
80	0	60	0	40	Yes
40	0	-20	40	0	No
20	60	0	0	-20	No
40	20	0	20	0	Yes
40	40	20	0	0	Yes

For each possibility, we try to solve the remaining three basic variables. Note that when x_1 and x_5 are chosen to be nonbasic, we can not find any solution that satisfies constraint $x_1 - x_5 = 40$ (the entries "N/S" means "no solution").

(b) According to Part (a), the four bfs are $(50, 0, 0, 30, 10)$, $(80, 0, 60, 0, 40)$, $(40, 20, 0, 20, 0)$, and $(40, 40, 20, 0, 0)$. Each of them corresponds to an extreme point shown in Figure 1.

bfs	Extreme point
$(50, 0, 0, 30, 10)$	$(50, 0)$
$(80, 0, 60, 0, 40)$	$(80, 0)$
$(40, 20, 0, 20, 0)$	$(40, 20)$
$(40, 40, 20, 0, 0)$	$(40, 40)$

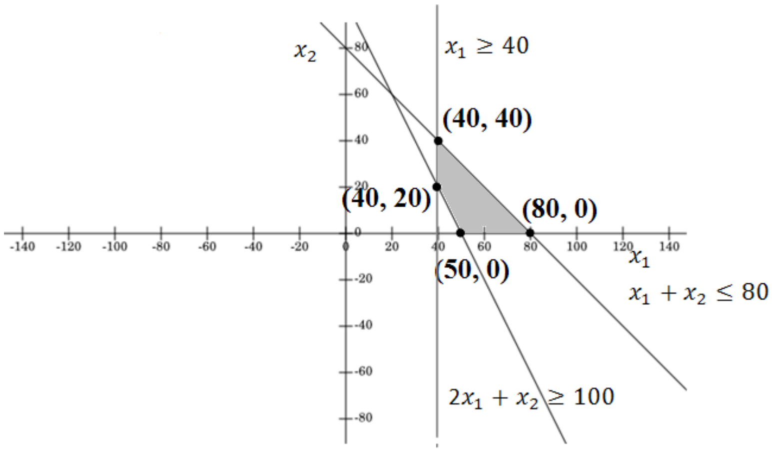

Figure 1: Extreme points for Problem 2b

3. The initial tableau is

-2	1	1	0	0	0	0
3	1	1	1	0	0	60
1	-1	2	0	1	0	10
1	1	-1	0	0	1	20

We run two iterations to get

-2	1	1	0	0	0	0
3	1	1	1	0	0	60
1	-1	2	0	1	0	10
1	$\boxed{1}$	-1	0	0	1	20

$\rightarrow$

-3	0	2	0	0	-1	-20
2	0	$\boxed{2}$	1	0	-1	40
2	0	1	0	1	1	30
1	1	-1	0	0	1	20

$\rightarrow$

-5	0	0	-1	0	0	-60
1	0	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	20
1	0	0	$-\frac{1}{2}$	1	$\frac{3}{2}$	10
2	1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	40

An optimal solution to the original LP is $(x_1^*, x_2^*, x_3^*) = (0, 40, 20)$ with objective value $z^* = -60$.

4. (a) The initial tableau is

-3	-2	0	0	0	0
2	1	1	0	0	100
1	1	0	1	0	80
1	0	0	0	1	40

We run three iterations to get

$$
\begin{array}{ccccc|c}
-3 & -2 & 0 & 0 & 0 & 0 \\
\hline
2 & 1 & 1 & 0 & 0 & 100 \\
1 & 1 & 0 & 1 & 0 & 80 \\
\boxed{1} & 0 & 0 & 0 & 1 & 40
\end{array}
\rightarrow
\begin{array}{ccccc|c}
0 & -2 & 0 & 0 & 3 & 120 \\
\hline
0 & \boxed{1} & 1 & 0 & -2 & 20 \\
0 & 1 & 0 & 1 & -1 & 40 \\
1 & 0 & 0 & 0 & 1 & 40
\end{array}
$$

$$
\rightarrow
\begin{array}{ccccc|c}
0 & 0 & 2 & 0 & -1 & 160 \\
\hline
0 & 1 & 1 & 0 & -2 & 20 \\
0 & 0 & -1 & 1 & \boxed{1} & 20 \\
1 & 0 & 0 & 0 & 1 & 40
\end{array}
\rightarrow
\begin{array}{ccccc|c}
0 & 0 & 1 & 1 & 0 & 180 \\
\hline
0 & 1 & -1 & 2 & 0 & 60 \\
0 & 0 & -1 & 1 & 1 & 20 \\
1 & 0 & 1 & -1 & 0 & 20
\end{array}
$$

An optimal solution to the original LP is $(x_1^*, x_2^*) = (20, 60)$ with objective value $z^* = 180$.

(b) The route we go through is depicted in Figure 2. We start from $(0, 0)$ and go through $(40, 0)$ and $(40, 20)$; finally we arrive at (20, 60).

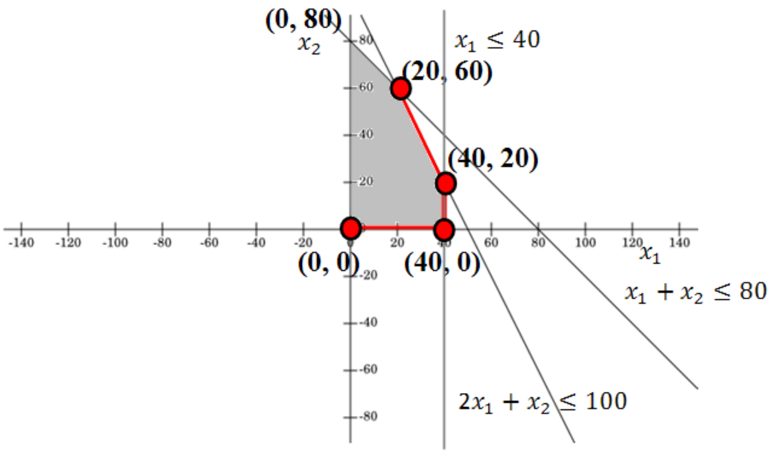

Figure 2: Graphical solution for Problem 4.b

5. (a) When we solve a maximization LP, we aim to increase the objective value. When we enter a variable with a negative reduced cost, increasing it leads the objected value to increase to balance the equity. For example, suppose the objective row is $z - x_1 = 0$, in which x_1 has a negative reduced cost. When x_1 increases, in order to maintain the equality, z must increase.

   (b) In running the simplex method, we start from an extreme point and move along an edge to another extreme point. The minimum ratio will help us find the first constraint we hit. On the other hand, if we choose a variable with a larger ratio, we will end up with a negative RHS, which makes the solution infeasible.

6. (a) The LP is

$$
\begin{array}{rl}
\max & 9x_1 + 50x_2 + 100x_3 \\
\text{s.t.} & x_1 + 4x_2 + 7x_3 \leq 40 \\
& x_i \geq 0 \quad \forall i = 1, 2, 3.
\end{array}
$$

(b) The LP is

$$\begin{aligned} \max \quad & 9y_1 + 32y_2 + 50y_3 \\ \text{s.t.} \quad & y_1 + 2y_2 + 3y_3 \leq 40 \\ & y_1 - 2y_2 \geq 0 \\ & y_2 - y_3 \geq 0 \\ & y_i \geq 0 \quad \forall i = 1, 2, 3. \end{aligned}$$

(c) As we we know the sales quantity of an item is equal to its production quantity minus the amount used for producing other products, we have $y_1 - 2y_2 = x_1$, $y_2 - y_3 = x_2$, $y_3 = x_3$. Therefore, we may transform each LP to the other as follows:

$$\begin{array}{ccc} 9x_1 + 50x_2 + 100x_3 & \Leftrightarrow & 9y_1 + 32y_2 + 50y_3 \\ x_1 + 4x_2 + 7x_3 \leq 40 & \Leftrightarrow & y_1 + 2y_2 + 3y_3 \leq 40 \\ x_1 \geq 0 & \Leftrightarrow & y_1 - 2y_2 \geq 0 \\ x_2 \geq 0 & \Leftrightarrow & y_2 - y_3 \geq 0 \\ x_3 \geq 0 & \Leftrightarrow & y_3 \geq 0 \end{array}$$