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Suggested Solution for Homework 4

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1. (a) The standard form is

$$\begin{array}{ll} \min & x_7 \\ \text{s.t.} & x_1 + 2x_2 + x_3 + x_4 = 2 \\ & x_1 + 2x_2 + x_5 = 3 \\ & x_1 + 2x_2 + 3x_3 - x_6 + x_7 = 3 \\ & x_i \geq 0 \quad \forall i = 1, ..., 7. \end{array}$$

The iterations for phase I are as follows:

$$\begin{array}{ccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 1 & 2 & 1 & 1 & 0 & 0 & 0 & 2\ (x_4) \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 & 3\ (x_5) \\ 1 & 2 & 3 & 0 & 0 & -1 & 1 & 3\ (x_7) \end{array} \rightarrow \begin{array}{ccccccc|c} 1 & 2 & 3 & 0 & 0 & -1 & 0 & 3 \\ \hline \boxed{1} & 2 & 1 & 1 & 0 & 0 & 0 & 2\ (x_4) \\ 1 & 2 & 0 & 0 & 1 & 0 & 0 & 3\ (x_5) \\ 1 & 2 & 3 & 0 & 0 & -1 & 1 & 3\ (x_7) \end{array}$$

$$\rightarrow \begin{array}{ccccccc|c} 0 & 0 & 2 & -1 & 0 & -1 & 0 & 1 \\ \hline 1 & 2 & 1 & 1 & 0 & 0 & 0 & 2\ (x_1) \\ 0 & 0 & -1 & -1 & 1 & 0 & 0 & 1\ (x_5) \\ 0 & 0 & \boxed{2} & -1 & 0 & -1 & 1 & 1\ (x_7) \end{array} \rightarrow \begin{array}{ccccccc|c} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ \hline 1 & 2 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & \frac{-1}{2} & \frac{3}{2}\ (x_1) \\ 0 & 0 & 0 & \frac{-3}{2} & 1 & \frac{-1}{2} & \frac{1}{2} & \frac{3}{2}\ (x_5) \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{-1}{2} & \frac{1}{2} & \frac{1}{2}\ (x_3) \end{array}.$$

(b) The iterations for phase II are as follows:

$$\begin{array}{cccccc|c} -4 & -4 & -1 & 0 & 0 & 0 & 0 \\ \hline 1 & 2 & 0 & \frac{3}{2} & 0 & \frac{1}{2} & \frac{3}{2}\ (x_1) \\ 0 & 0 & 0 & \frac{-3}{2} & 1 & \frac{-1}{2} & \frac{3}{2}\ (x_5) \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{-1}{2} & \frac{1}{2}\ (x_3) \end{array} \rightarrow \begin{array}{cccccc|c} 0 & 4 & 0 & \frac{11}{2} & 0 & \frac{3}{2} & \frac{13}{2} \\ \hline 1 & \boxed{2} & 0 & \frac{3}{2} & 0 & \frac{1}{2} & \frac{3}{2}\ (x_1) \\ 0 & 0 & 0 & \frac{-3}{2} & 1 & \frac{-1}{2} & \frac{3}{2}\ (x_5) \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{-1}{2} & \frac{1}{2}\ (x_3) \end{array}$$

$$\rightarrow \begin{array}{cccccc|c} -2 & 0 & 0 & \frac{5}{2} & 0 & \frac{1}{2} & \frac{7}{2} \\ \hline \frac{1}{2} & 1 & 0 & \boxed{3/4} & 0 & \frac{1}{4} & \frac{3}{4}\ (x_2) \\ 0 & 0 & 0 & \frac{-3}{2} & 1 & \frac{-1}{2} & \frac{3}{2}\ (x_5) \\ 0 & 0 & 1 & \frac{-1}{2} & 0 & \frac{-1}{2} & \frac{1}{2}\ (x_3) \end{array} \rightarrow \begin{array}{cccccc|c} \frac{-11}{3} & \frac{-10}{3} & 0 & 0 & 0 & \frac{-1}{3} & 1 \\ \hline \frac{2}{3} & \frac{4}{3} & 0 & 1 & 0 & \frac{1}{3} & 1\ (x_4) \\ 1 & 2 & 0 & 0 & 1 & 0 & 3\ (x_5) \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 & 0 & \frac{-1}{3} & 1\ (x_3) \end{array}.$$

The optimal solution to the original LP is $(x_1^*, x_2^*, x_3^*) = (0, 0, 1)$ with objective value $z^* = 1$.

2. (a) The iterations using the smallest index rule are as follows:

$$\begin{array}{ccccc|c} 0 & 2 & 0 & 4 & 0 & 10 \\ \hline 0 & 3 & 0 & 2 & 1 & 6 \\ 0 & \boxed{1} & 1 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 3 \end{array} \rightarrow \begin{array}{ccccc|c} 0 & 0 & -2 & 2 & 0 & 6 \\ \hline 0 & 0 & -3 & -1 & 1 & 0 \\ 0 & 1 & 1 & \boxed{1} & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 3 \end{array} \rightarrow \begin{array}{ccccc|c} 0 & -2 & -4 & 0 & 0 & 2 \\ \hline 0 & 1 & -2 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 & 0 & 2 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}.$$

The optimal bfs is $(x_1^*, x_2^*, x_3^*, x_4^*, x_5^*) = (1, 0, 0, 2, 2)$ with objective value $z^* = 2$.

(b) If there exists one basic variable being 0 in one of all the iterations, we say that the LP is degenerate. In iteration 1 in part (a), the basic variable $x_5 = 0$, so the LP is degenerate.

(c) No.

3. (a) $c_1 \geq 0, c_2 \geq 0$.

   (b) $c_2 < 0, a_1 \leq 0$.

   (c) $b = 0$.

   (d) $c_1 < 0, a_3 b \geq 12$.

   (e) $c_1 \geq 0, c_2 \geq 0, c_1 c_2 = 0$.

   (f) $c_1 \geq 0, c_2 = 0, a_1 \leq 0$.

4. (a) Because the constraint $w \geq \max\{x_1, x_2\}$ will be binding at any optimal solution for the minimization problem, w will be equal to $\max\{x_1, x_2\}$.

   (b) Because $\max\{x_1, x_2\}$ is to choose the bigger one between x_1 and x_2, w must be greater or equal to both x_1 and x_2, i.e., $w \geq x_1$ and $w \geq x_2$. Moreover, as long as $w \geq x_1$ and $w \geq x_2$, w will be greater than or equal to $\max\{x_1, x_2\}$. Therefore, $w \geq \max\{x_1, x_2\}$ is equivalent to $w \geq x_1$ and $w \geq x_2$.

5. Let our decision variables be

$$\begin{aligned} x &= \text{the } x\text{-coordinate of the station and} \\ y &= \text{the } y\text{-coordinate of the station} \end{aligned}$$

Denote (A_i, B_i) as the location of city i and F_i as the average number of fires in city i, $i = 1, ..., 5$. A nonlinear formulation is

$$\min \quad \sum_{i=1}^{5} F_i \Big(|x - A_i| + |y - B_i| \Big),$$

which is equivalent to

$$\begin{aligned} \min \quad & \sum_{i=1}^{5} F_i \Big(u_i + v_i \Big) \\ \text{s.t.} \quad & u_i \geq |x - A_i| \quad \forall i = 1, ..., 5 \\ & v_i \geq |y - B_i| \quad \forall i = 1, ..., 5 \\ & x, y, u_i, y_i \geq 0 \quad \forall i = 1, ..., 5, \end{aligned}$$

because the constraints $u_i \geq |x - A_i|$ and $v_i \geq |y - B_i|$ will be binding at any optimal solution. The above formulation can be further linearized into

$$\begin{aligned} \min \quad & \sum_{i=1}^{5} F_i \Big(u_i + v_i \Big) \\ \text{s.t.} \quad & u_i \geq x - A_i \quad \forall i = 1, ..., 5 \\ & u_i \geq A_i - x \quad \forall i = 1, ..., 5 \\ & v_i \geq y - B_i \quad \forall i = 1, ..., 5 \\ & v_i \geq B_i - y \quad \forall i = 1, ..., 5 \\ & x, y, u_i, v_i \geq 0 \quad \forall i = 1, ..., 5. \end{aligned}$$

6. I have tried my best to read the textbook in the past seven days.