Operations Research, Spring 2014
Homework 6

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

Note 1. The deadline of this homework is ***1pm, April 17, 2014***. Please put a hard copy of the work into the instructor's mailbox on the first floor of the Management Building II by the due time. Late submissions will not be accepted. Each student must submit her/his individual work.

Note 2. In total there are 120 points for this homework.

1. (20 points; 5 points each) Consider the network

$$G = (V, E) = \left(\left\{1, 2, 3, 4, 5\right\}, \left\{(1,2), (2,3), (2,4), (3,1), (3,4), (4,5), (5,2)\right\}\right).$$

   (a) Draw a figure to visualize it.

   (b) Find all the cycles.

   (c) Suppose that the distances of these arcs are provided below:

(i, j)	$(1,2)$	$(2,3)$	$(2,4)$	$(3,1)$	$(3,4)$	$(4,5)$	$(5,2)$
weight	8	3	4	3	2	4	3

   Find the shortest path from 1 to 5.

   (d) If the above weights are capacities, find the maximum flow from 1 to 5.

2. (20 points; 10 points each) A product is produced in two factories and sold in five markets. The unit production costs of factories 1 and 2 are \$10 and \$12, respectively. The unit retail price of markets 1, 2, 3, 4, and 5 are \$15, \$20, \$18, \$21, and \$25, respectively. The unit shipping costs from factories to markets are provided below:

Factory	Market				
	1	2	3	4	5
1	\$2	\$3	\$3	\$2	\$1
2	\$1	\$2	\$2	\$1	\$2

   The capacities of factories 1 and 2 are 1000 and 1500 units, respectively. The demands of markets 1, 2, 3, and 4 are 400, 700, 500, 100, and 600 units, respectively. A plan of producing, shipping, and selling products that maximizes the total profits is desired.

   (a) Graphically formulate the problem as a transportation problem.

   (b) Mathematically formulate the problem as a transportation problem.

3. (15 points) Five kids are competing for three cakes. The levels of happiness for kid i to eat cake j, h_{ij}s, are recorded below. For example, $h_{12} = 5$. Cakes cannot be divided. One kid can eat at most one cake. All cakes must be assigned. If a kid does not get a cake, her/his level of happiness is 5.

Kid	Cake		
	1	2	3
1	9	5	6
2	7	8	3
3	4	7	5
4	3	6	1
5	2	8	4

(a) (10 points) Formulate the problem of assigning cakes in a way that maximizes the average happiness as an assignment problem.

(b) (5 points) How does your answer in Part (a) change if the level of happiness is 0 when one gets no cake?

4. (20 points) Among a set of experts $I = \{1, 2, 3, 4, 5, 6\}$, some of them are to be selected and brought to a foreign country. Bringing expert i incurs a cost c_i. Once a group of experts S_j are selected, mission $j \in J = \{1, 2, 3, 4\}$ can be executed and a benefit b_j can be collected. The following table lists the experts required for each project. An expert will be paid only once no matter how many missions she/he works for.

Mission	1	2	3	4
Experts	$\{1, 2, 5\}$	$\{1, 4, 6\}$	$\{2, 3, 4, 5\}$	$\{3, 6\}$

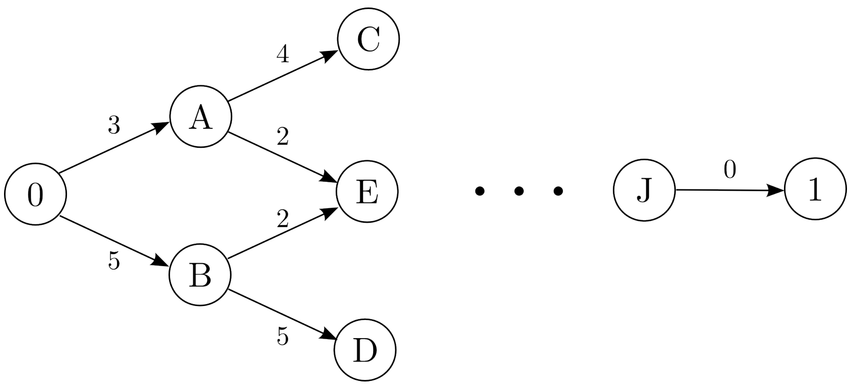
The problem is to select some experts to maximize the total profits. Formulate an IP whose LP relaxation gives an integer solution by the simplex method that solves this problem.

Note. You do not need to prove that the LP relaxation really gives an integer solution. You will get full credits as long as your IP does have the desired property.

5. (15 points) To complete a project, ten tasks, each with a processing time, must be done. Tasks may be processed in parallel. However, a task may be started only after the completion of some other tasks (i.e., a task may have some "prerequisites"). The following table shows the processing times and prerequisites, if any, of each job.

Task	A	B	C	D	E	F	G	H	I	J
Processing time	3	5	4	5	2	6	8	5	4	6
Prerequisites	$\emptyset$	$\emptyset$	$\{A\}$	$\{B\}$	$\{A, B\}$	$\{C, E\}$	$\{D\}$	$\{F\}$	$\{G, H\}$	$\{I\}$

Given these ten tasks, a natural question is: How long does it take to complete the whole project if all tasks are started as soon as possible? To answer this problem, let's start by visualizing the relationship among tasks:



In this figure, node i represents task i while c_{ij} of arc (i, j) represents the processing time of task j. If we complete the figure and find a *longest path* from node 0 (which means "starting") to node 1 (which means "ending"), the length of the path will tell us the desired answer.

(a) (5 points) Complete this figure and find a longest path.

(b) (10 points) Find the starting and completion times of tasks A, E, and J.

6. (10 basic points and 10 bonus points) Continue from the previous problem. When the number of tasks and precedence relations are large enough, we need the help of mathematical programming for solving the problem. Let

$$x_i = \text{the completion time of task } i, i = 0, A, B, ..., I, 1,$$

a constraint that relates tasks 0 and A is $x_A = x_0 + 3$. Similarly, a constraint that relates tasks A and C is $x_C = x_A + 4$. But how about tasks A, B, and E?

(a) (5 points) Write down two constraints that relate tasks A, B, and E.

(b) (5 points) Write down all constraints that relate tasks 0, A, B, ..., and E.

(c) (10 bonus points) Write down the compact formulation of this problem in general.
Hint. Do not forget to set x_0 to 0.

7. (10 bonus points) Continue from the previous problem. Now we have an LP formulation for this problem. As long as we solve the LP, we know the completion times of all tasks. For this "problem on a network" which is not an MCNF problem, will the simplex method always reports an integer solution (if a fractional solution is reported, it is very hard to interpret it!)? Show that the coefficient matrix of this problem is totally unimodular.