Operations Research, Spring 2014
Suggested Solution for Homework 3

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1. (a) The figure of network is

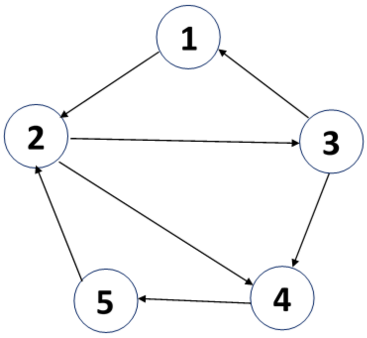

Figure 1: Network for Problem 1a

(b) There are three independent cycles in the network.

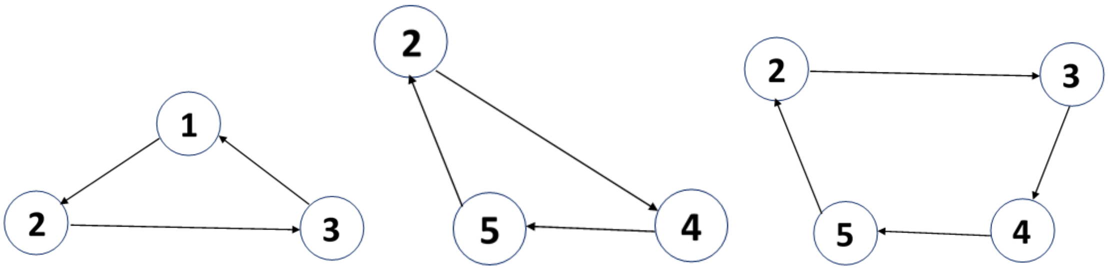

Figure 2: Cycles in network for Problem 1b

(c) The shortest path from 1 to 5 is

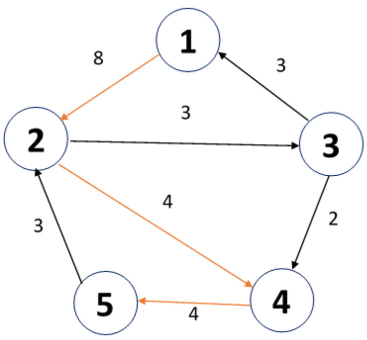

Figure 3: Shortest path from 1 to 5 for Problem 1c

The total distance is 16.

(d) The maximum flow from 1 to 5 is shown in Figure 4. For each arc, the label means its capacity and flow size.

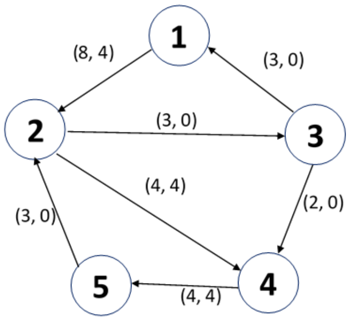

Figure 4: maximum flow from 1 to 5 for Problem 1d

2. (a) The problem can be graphically formulated as Figure 5.

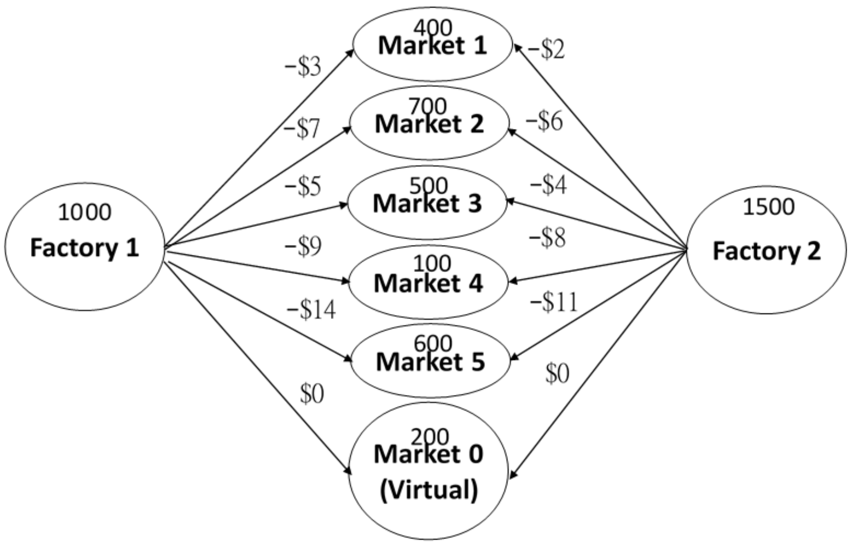

Figure 5: Graphically formulate as a transportation problem for Problem 2a

Since the supply is larger than demand, there exists a virtual market to balance.

(b) We define

$$
\begin{aligned}
x_{ij} &= \text{unit of product from factory } i \text{ to market } j \ , i = 1, 2, j = 0, ..., 5 \\
C^s_{ij} &= \text{unit shipping costs from factory } i \text{ to market } j \ , i = 1, 2, j = 0, ..., 5 \\
C^p_i &= \text{unit production costs of factory } i \ , i = 1, 2 \\
C^r_j &= \text{unit retail price of market } j \ , j = 1, ..., 5 \\
S_i &= \text{supply of factory } i \ , i = 1, 2 \\
D_j &= \text{demand of market } j \ , j = 1, ..., 5.
\end{aligned}
$$

The market 0 is a virtual market and its virtual demand is 200 units. The formulation is

$$\begin{aligned}
\min \quad & \sum_{i=1}^{2}\sum_{j=1}^{5}(C^s_{ij} + C^p_i - C^r_j)x_{ij} \\
\text{s.t.} \quad & \sum_{j=0}^{5} x_{ij} = S_i \quad \forall i = 1, 2 \\
& \sum_{i=1}^{2} x_{ij} = D_i \quad \forall j = 1, ..., 5 \\
& x_{ij} \geq 0 \quad \forall i = 1, 2 \quad \forall j = 0, ..., 5.
\end{aligned}$$

The objective function minimizes the total costs. The first and second constraints ensure the supply and demand.

3. (a) We define

$$\begin{aligned}
x_{ij} &= \begin{cases} 1 & \text{if kid } i \text{ eat cake } j \\ 0 & \text{otherwise} \end{cases} , i = 1, ..., 5, j = 1, ..., 5 \\
h_{ij} &= \text{ happiness level when kid } i \text{ eats cake } j \ , i = 1, ..., 5, j = 1, ..., 5.
\end{aligned}$$

x_{ij} is a decision variable and h_{ij} is a parameter. We make two virtual cakes so that each kid can get one cake. Virtual cake can lead 5 happiness level for a kid ($h_{ij} = 5$,$i = 1, ..., 5, j = 4, 5$). The formulation is

$$\begin{aligned}
\min \quad & \sum_{j=1}^{5}\sum_{i=1}^{5} -(h_{ij}x_{ij}) \\
\text{s.t.} \quad & \sum_{j=1}^{5} x_{ij} = 1 \quad \forall i = 1, ..., 5 \\
& \sum_{i=1}^{5} x_{ij} = 1 \quad \forall j = 1, ..., 5 \\
& x_{ij} \in 0, 1 \quad \forall i = 1, ..., 5 \quad \forall j = 1, ..., 5.
\end{aligned}$$

The first constraint ensures that each kid get a cake. The second constraint ensures that each cake is assigned to a kid.

(b) We replace the value of parameter h_{ij} by $h_{ij} = 0 \quad \forall i = 1, ..., 5 \quad \forall j = 4, 5$. The formulation is identical to Part (a).

4. We define

$$\begin{aligned}
x_i &= \begin{cases} 1 & \text{if expert } i \text{ is selected} \\ 0 & \text{otherwise} \end{cases} , i = 1, ..., 6 \\
S_j &= \begin{cases} 1 & \text{if mission } j \text{ is executed} \\ 0 & \text{otherwise} \end{cases} , j = 1, ..., 4 \\
U &= \{(1,1), (1,2), (1,5), (2,1), (2,4), (2,6), (3,2), (3,3), (3,4), (3,5), (4,3), (4,6)\}.
\end{aligned}$$

x_i, S_j are decision variables and h_{ij} is a parameter. Element (j, i) in U means job j needs expert i to complete. The formulation is

$$\begin{aligned}
\min \quad & \sum_{i=1}^{6} c_i x_i - \sum_{j=1}^{4} S_j b_j \\
\text{s.t.} \quad & x_i \geq S_j \quad \forall (j, i) \in U \\
& x_i \in 0, 1 \quad \forall i = 1, ..., 6 \\
& S_j \in 0, 1 \quad \forall j = 1, ..., 4.
\end{aligned}$$

The objective function minimizes the total cost. The first constraint ensures that the job will be executed when all of the experts for the job are selected.

5. (a) The complete figure in this problem is

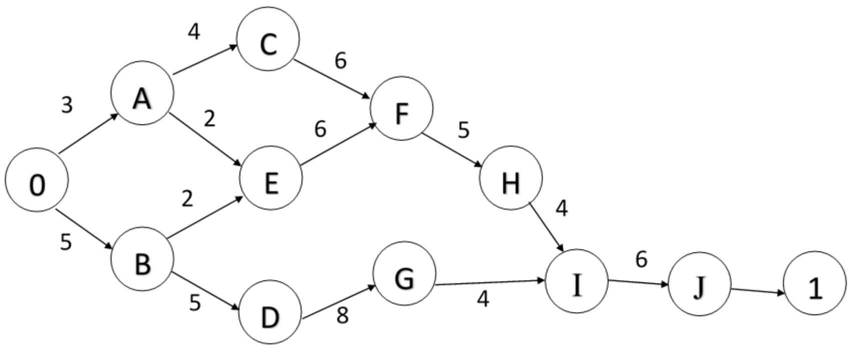

Figure 6: Network for Problem 5a

There exits three longest paths (1) 0 → A → C → F → H → I → J → 1 (2) 0 → B → D → G → I → J → 1 (3) 0 → B → E → F → H → I → J → 1; The distance is 28.

(b) The starting and completion time of A, E, and J are as following

	starting time	complete time
A	0	3
E	5	7
J	22	28

6. (a) The constraints that relate tasks A, B, and E are

$$x_A + 2 \leq x_E$$

$$x_B + 2 \leq x_E$$

(b) The constraints that relate tasks 0, A, B,..., E are

$$x_0 = 0$$

$$x_0 + 3 = x_A$$

$$x_0 + 5 = x_B$$

$$x_A + 4 = x_C$$

$$x_B + 5 = x_D$$

$$x_A + 2 \leq x_E$$

$$x_B + 2 \leq x_E$$

(c) The formulation is

$$\begin{array}{rl}
\min & x_1 \\
\text{s.t.} & x_0 + 3 = x_A \\
& x_0 + 5 = x_B \\
& x_A + 4 = x_C \\
& x_B + 5 = x_D \\
& x_A + 2 \leq x_E \\
& x_B + 2 \leq x_E \\
& x_C + 6 \leq x_F \\
& x_E + 6 \leq x_F \\
& x_D + 8 = x_G \\
& x_F + 5 = x_H \\
& x_G + 4 \leq x_I \\
& x_H + 4 \leq x_I \\
& x_I + 6 = x_J \\
& x_i \geq 0 \quad \forall i = 0, A, B, ..., J, 1.
\end{array}$$

7. We define the origin coefficient matrix with A, the transportation matrix is A^t. Since it is obvious that A^t is totally unimodular, the origin coefficient matrix will be totally unimodular as well.