Operations Research, Spring 2014
Suggested Solution for Homework 9

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. (a) If player 2 plays L, player 1's best response is T; if player 2 plays C, player 1's best response is M; if player 2 plays R, player 1's best response is B.

   (b) If player 1 plays T, player 2's best response is L; if player 1 plays M, player 2's best response is C; if player 1 plays B, player 2's best response is L.

   (c) For player 2, R is strictly dominated by C. Please note that R is not strictly dominated by L; it is only weakly dominated by L because L and R both result in 1 as the payoff under M.

   (d) (T, L) and (M, C) are Nash equilibria.

2. (a) No, because if I know the opponent chooses 50, my best response is to choose any integer below 50, which makes me the only winner.

   (b) No, because if I know the two opponents choose 33 and 33, my best response is to choose any integer below 33, which makes me the only winner.

   (c) The only Nash equilibrium is $(x_1, x_2) = (1, 1)$. For any $(x_1, x_2) \neq (1, 1)$, the player not choosing a smaller number has an incentive to decrease the number she chooses. Therefore, $(1, 1)$ is the unique Nash equilibrium.

   (d) The only Nash equilibrium is $(x_1, ..., x_{10}) = (1, ..., 1)$. By the same argument used in Part (c), it is a unique Nash equilibrium.

3. (a) For village 1, it has no incentive to invest more because that simply wastes money. It also has no incentive to invest less because that will make the fund insufficient for building the bridge. As the situation is the same for village 2, no one wants to unilaterally deviate. Therefore, $(4000, 4000)$ is a Nash equilibrium. On the other hand, $(0, 0)$ is not a Nash equilibrium because if one village knows the other would invest 0, its best response is to invest 8000.

   (b) If $x_1 + x_2 > 8000$, either village has an incentive to decrease its investment as long as $x_1 + x_2 \geq 8000$; if $x_1 + x_2 < 8000$, either village has an incentive to increase its investment to make $x_1 + x_2 = 8000$. Therefore, a combination of x_1 and x_2 is a Nash equilibrium if and only if $x_1 + x_2 = 8000$.

   (c) For village 1, it has no incentive to invest more because that simply wastes money. It also has no incentive to invest less because that will make the fund insufficient for building the bridge. As the situation is the same for village 2, no one wants to unilaterally deviate. Therefore, $(4000, 4000)$ is a Nash equilibrium. Moreover, $(0, 0)$ is also a Nash equilibrium because if one village knows the other would invest 0, its best response is to invest nothing.

   (d) If $x_1 + x_2 > 12000$, either village has an incentive to decrease its investment as long as $x_1 + x_2 \geq 12000$. If $x_1 + x_2 = 12000$, no village has an incentive to deviate. Finally, consider $x_1 + x_2 < 12000$. If $x_2 \geq 2000$, player 1 has an incentive to deviate and make $x_1 + x_2 = 12000$. If $x_2 < 2000$, player 1 has an incentive to deviate to invest nothing. Therefore, if $x_1 + x_2 < 12000$, the only Nash equilibrium is $(0, 0)$. Collectively, the set of Nash equilibria is $\{(x_1, x_2) | x_1 + x_2 = 12000\} \cup \{(0, 0)\}$.

4. (a) For firm 1, it acts to maximize its profit

$$\min_{q_1} \ \pi_1 = (a - q_1 - q_2 - q_3 - c)q_1.$$

   By the FOC, we get

$$q_1 = \frac{a - q_2 - q_3 - c}{2}.$$

Similar for firm 2 and 3, we get

$$q_2 = \frac{a - q_1 - q_3 - c}{2} \quad \text{and} \quad q_3 = \frac{a - q_1 - q_2 - c}{2}.$$

By solving the above three equalities, we get the Nash equilibrium as (q_1^*, q_2^*, q_3^*), which satisfies

$$q_1^* = q_2^* = q_3^* = \frac{1}{4}(a - c).$$

(b) For firm 1, it acts to maximize its profit

$$\min_{q_1} \ \pi_1 = (a - q_1 - q_2 - \cdots - q_n - c)q_1.$$

By the FOC, we get

$$q_1 = \frac{a - q_2 - q_3 - \cdots - q_n - c}{2}.$$

Similar for firm 2 to n, we get

$$\begin{aligned} q_2 &= \frac{a - q_1 - q_3 - \cdots - q_n - c}{2}, \\ q_3 &= \frac{a - q_1 - q_2 - q_4 - \cdots - q_n - c}{2}, \cdots, \text{ and} \\ q_n &= \frac{a - q_1 - q_2 - \cdots - q_{n-1} - c}{2}. \end{aligned}$$

By solving the above three equalities, we get the Nash equilibrium as $(q_1^*, ..., q_n^*)$, which satisfies

$$q_1^* = q_2^* = \cdots = q_n^* = \frac{1}{n+1}(a - c).$$

(c) The equilibrium quantity chosen by a single firm is $\frac{1}{n+1}(a-c)$, which decreases in n. Intuitively, when the number of firms increases, the market-clearing price decreases, and a firm must cut down the supply quantity to respond to the decreased price.

(d) Since a single firm's equilibrium quantity and the market price both decrease in n, the equilibrium profit earned by a single firm decreases in n. Intuitively, when there are more competitors, a firm will earn less money.

5. If they do not cooperate, the optimal prices are $p_1^D = p_2^D = \frac{a+c}{2-b}$. If they cooperate, the optimal prices are $p_1^I = p_2^I = \frac{a+c(1-b)}{2(1-b)}$. We can show that

$$p_1^D - p_1^I = \frac{a + c}{2 - b} - \frac{a + c(1 - b)}{2(1 - b)} = -[b(a - c) + b^2 c] < 0.$$

Therefore, $p_1^D < p_1^I$ and integration hurts consumers due to higher equilibrium prices. For firms' profits, the total profit earned under decentralization is

$$\pi^D = 2\left(\frac{a + c}{2 - b} - c\right)\left(a - \frac{a + c}{2 - b} + b \cdot \frac{a + c}{2 - b}\right) = 2\left(\frac{a - c + bc}{2 - b}\right)^2.$$

Moreover, the total profit earned under integration is

$$\begin{aligned} \pi^I &= 2\left(\frac{a + c(1 - b)}{2(1 - b)} - c\right)\left(a - \frac{a + c(1 - b)}{2(1 - b)} + b \cdot \frac{a + c(1 - b)}{2(1 - b)}\right) \\ &= 2\left(\frac{a + c(1 - b)}{2(1 - b)}\right)\left(a - (1 - b) \cdot \frac{a + c(1 - b)}{2(1 - b)}\right) = \frac{(a - c + bc)^2}{2(1 - b)}. \end{aligned}$$

Therefore, we have

$$\pi^I - \pi^D = (a-c+bc)^2\left(\frac{1}{2(1-b)} - \frac{2}{(2-b)^2}\right)$$
$$= \frac{(a-c+bc)^2}{2(1-b)(2-b)^2}\Big[(2-b)^2 - 4(1-b)\Big] = \frac{(a-c+bc)^2}{2(1-b)(2-b)^2}\cdot b^2 > 0.$$

Since $\pi^I > \pi^D$, integration benefits firms due to higher aggregate equilibrium profit.

6. For firm 1, it acts to maximize its profit

$$\min_{q_1} \pi_1 = (a_1 - p_1 + bp_2)(p_1 - c_1).$$

By the FOC, we get $p_1 = \frac{a_1+bp_2+c_1}{2}$. For firm 2, it also acts to maximize its profit by choosing $p_2 = \frac{a_2+bp_1+c_2}{2}$. By solving the above two equations, we get the Nash equilibrium (p_1^*, p_2^*), which satisfies

$$p_1^* = \frac{a_2b + 2a_1 + bc_2 + 2c_1}{4-b^2} \quad \text{and} \quad p_2^* = \frac{a_1b + 2a_2 + bc_1 + 2c_2}{4-b^2}.$$

(a) Given $c_1 = c_2$ and $a_1 > a_2$, we have

$$p_1^* - p_2^* = \frac{1}{2+b}(a_1 - a_2) > 0.$$

Therefore, we prove that firm 1 always chooses a higher price than firm 2 in equilibrium. The statement is the problem is true.

(b) Given $c_1 > c_2$ and $a_1 = a_2$,

$$p_2^* - p_1^* = \frac{1}{4-b^2}(c_1 - c_2)(b-2) < 0.$$

Therefore, we prove that firm 1 always chooses a higher price than firm 2 in equilibrium. The statement in the problem is false.