IM 2010: Operations Research, Spring 2014

The Simplex Method (Part 2)

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Introduction

- Last time we introduced the simplex method.
- There remain some unsolved problem:
  - How to find an initial bfs? How to know whether an LP is infeasible?
  - What if an LP is unbounded?
  - What if multiple nonbasic variables may be entered?
  - What if there is a tie in a ratio test?
  - How efficient the simplex method is?
- In this lecture, we will address these issues (and some more).
- Read Sections 4.5 and 4.6 thoroughly.
  - Sections 4.8 and 4.9 contain discussions regarding efficiency.

Road map

- **Information on tableaus**.
- Finding an initial bfs.
- Degeneracy and efficiency.
- The matrix way of doing simplex.

Identifying unboundedness

- When is an LP **unbounded**?
- An LP is unbounded if:
  - There is an improving direction.
  - Along that direction, we may move forever.
- When we run the simplex method, this can be easily checked in a simplex tableau.
- Consider the following example:

$$\begin{array}{rrcrcl} \max & x_1 & & & & \\ \text{s.t.} & x_1 & - & x_2 & \leq & 1 \\ & 2x_1 & - & x_2 & \leq & 4 \\ & & x_i \geq 0 & \forall i = 1, 2. & & \end{array}$$

Unbounded LPs

- The standard form is:

$$\begin{array}{rrrrrrrrrr} \max & x_1 & & & & & & & & \\ \text{s.t.} & x_1 & - & x_2 & + & x_3 & & & = & 1 \\ & 2x_1 & - & x_2 & & & + & x_4 & = & 4 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.} \end{array}$$

- The first iteration:

$$\begin{array}{cccc|c} -1 & 0 & 0 & 0 & 0 \\ \hline \boxed{1} & -1 & 1 & 0 & x_3 = 1 \\ 2 & -1 & 0 & 1 & x_4 = 4 \end{array} \rightarrow \begin{array}{cccc|c} 0 & -1 & 1 & 0 & 1 \\ \hline 1 & -1 & 1 & 0 & x_1 = 1 \\ 0 & 1 & -2 & 1 & x_4 = 2 \end{array}$$

Unbounded LPs

- The second iteration:

0	-1	1	0	1
1	-1	1	0	$x_1 = 1$
0	$\boxed{1}$	-2	1	$x_4 = 2$

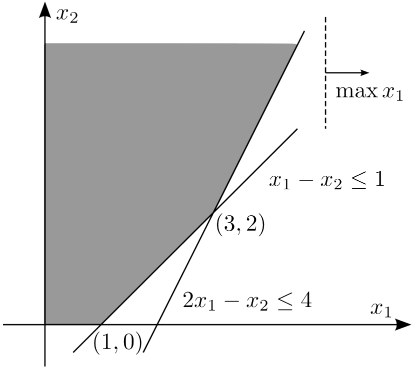
$\rightarrow$

0	0	-1	1	3
1	0	-1	1	$x_1 = 3$
0	1	-2	1	$x_2 = 2$

- How may we do the third iteration? The **ratio test** fails!
  - Only rows with positive denominators participate in the ratio test.
  - Now all the denominators are nonpositive! Which variable to leave?
- No one should leave: Increasing x_3 makes x_1 and x_2 become larger.
  - Row 1: $x_1 - x_3 + x_4 = 3$.
  - Row 2: $x_2 - 2x_3 + x_4 = 2$.
- The direction is thus an **unbounded improving direction**.

Unbounded improving directions

- At $(3, 2)$, when we enter x_3, we move along the rightmost edge. Geometrically, both nonbinding constraints $x_1 \geq 0$ and $x_2 \geq 0$ are "behind us".



Detecting unbounded LPs

- For a **minimization** LP, whenever we see **any** column in **any** tableau

$\bar{c}_j$	
d_1	
$\vdots$	
d_m	

  such that $\bar{c}_j > 0$ and $d_i \leq 0$ for all $i = 1, ..., m$, we may stop and conclude that this LP is unbounded.
  - $\bar{c}_j > 0$: This is an improving direction.
  - $d_i \leq 0$ for all $i = 1, ..., m$: This is an unbounded direction.
- What is the unbounded condition for a **maximization** problem?

Multiple optimal solutions

- Consider another example (in standard form directly):

$$
\begin{array}{rrcrcrcrcrcl}
\max & x_1 & + & x_2 & & & & & & & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & x_3 & & & & & = & 12 \\
 & 2x_1 & + & x_2 & & & + & x_4 & & & = & 12 \\
 & x_1 & + & x_2 & & & & & + & x_5 & = & 7 \\
 & \multicolumn{11}{l}{x_i \geq 0 \quad \forall i = 1, ..., 5.}
\end{array}
$$

Multiple optimal solutions

- In two iterations, we find an optimal solution:

$$
\begin{array}{ccccc|c}
-1 & -1 & 0 & 0 & 0 & 0 \\
\hline
1 & 2 & 1 & 0 & 0 & x_3 = 12 \\
\boxed{2} & 1 & 0 & 1 & 0 & x_4 = 12 \\
1 & 1 & 0 & 0 & 1 & x_5 = 7
\end{array}
\rightarrow
\begin{array}{ccccc|c}
0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 6 \\
\hline
0 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 & x_3 = 6 \\
1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & x_1 = 6 \\
0 & \boxed{\frac{1}{2}} & 0 & -\frac{1}{2} & 1 & x_5 = 1
\end{array}
$$

$$
\rightarrow
\begin{array}{ccccc|c}
0 & 0 & 0 & 0 & 1 & 7 \\
\hline
0 & 0 & 1 & 1 & -2 & x_3 = 3 \\
1 & 0 & 0 & 1 & -2 & x_1 = 5 \\
0 & 1 & 0 & -1 & 2 & x_2 = 2
\end{array}
$$

Multiple optimal solutions

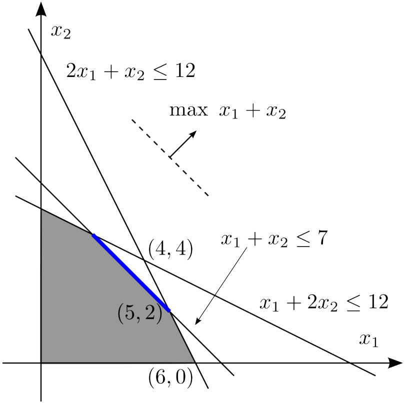
- In practice, we will simply stop and report the optimal solution.
- But here the optimal tableau shows the existence of **multiple** optimal solutions.

0	0	0	0	1	7
0	0	1	1	-2	$x_3 = 3$
1	0	0	1	-2	$x_1 = 5$
0	1	0	-1	2	$x_2 = 2$

- What does a zero reduced cost mean?
  - When we increase x_4, z will not be affected.
- As the current solution is optimal, if there is a direction such that moving along it does not change the objective value, **all points** along that direction are optimal.

Multiple optimal solutions

- At **an** optimal solution $(5, 2)$, by entering x_4, we move along $x_1 + x_2 = 7$. All points on that edge are optimal.
- For a nondegenerate LP, at an **optimal** tableau, if a nonbasic variable x_j has a **zero reduced cost**, the LP has multiple optimal solutions.
  - For a degenerate LP (to be discussed later in this lecture), the condition is not sufficient.
  - In practice, knowing this is not very valuable.



Road map

- Information on tableaus.
- **Finding an initial bfs**.
- Degeneracy and efficiency.
- The matrix way of doing simplex.

Feasibility of an LP

- When an LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

  satisfies $b \geq 0$, finding a bfs for its standard form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax + Iy = b \\ & x, y \geq 0, \end{aligned}$$

  is trivial.
  - We may form a feasible basis with all the slack variables y.
- What if there are some "=" or "$\geq$" constraints?

Feasibility of an LP

- For example, given an LP

$$\begin{array}{rrrrrrrrrr} \min & x_1 \\ \text{s.t.} & x_1 & + & x_2 & - & x_3 & + & x_4 & \geq & 10 \\ & 3x_1 & + & 2x_2 & + & 9x_3 & - & x_4 & = & 10 \\ & x_1 & - & 8x_2 & + & 2x_3 & - & 6x_4 & \leq & 10 \\ & x_i \geq 0 & \forall i = 1, ..., 4 \end{array}$$

whose standard form is

$$\begin{array}{rrrrrrrrrrrrrr} \min & x_1 \\ \text{s.t.} & x_1 & + & x_2 & - & x_3 & + & x_4 & - & x_5 & & & = & 10 \\ & 3x_1 & + & 2x_2 & + & 9x_3 & - & x_4 & & & & & = & 10 \\ & x_1 & - & 8x_2 & + & 2x_3 & - & 6x_4 & & & + & x_6 & = & 10 \\ & x_i \geq 0 & \forall i = 1, ..., 6, \end{array}$$

it is nontrivial to find a feasible basis (if there is one).

The two-phase implementation

- To find an initial bfs (or show that there is none), we may apply the **two-phase implementation**.
- Given a standard form LP (P), we construct a **phase-I LP** (Q):[1]

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \qquad\qquad (Q) \quad \begin{array}{ll} \min & 1^T y \\ \text{s.t.} & Ax + Iy = b \\ & x, y \geq 0. \end{array}$$

- (Q) has a trivial bfs $(x, y) = (0, b)$, so we can apply the simplex method on (Q). But so what?

Proposition 1

(P) is feasible if and only if (Q) has an optimal bfs $(x, y) = (\bar{x}, 0)$. In this case, $\bar{x}$ is a bfs of (P).

[1] Even if in (P) we have a maximization objective function, (Q) is still the same.

The two-phase implementation

- After we solve (Q), either we know (P) is infeasible or we have a feasible basis of (P).
- In the latter case, we can recover the objective function of the original (P) to get a **phase-II LP**.
  - "The phase-II LP" is nothing but the original (P).
  - Phase I for a **feasible** solution and phase II for an **optimal** solution.
- Regarding those added variables:
  - They are **artificial variables** and have no physical meaning. They are created only for checking feasibility.
  - If a constraint already has a variable that can be included in a trivial basis, we do not need to add an artificial variable in that constraint.
  - This happens to those "$\leq$" constraints (if the RHS is nonnegative).
- We then **adjust** the tableau according to the initial basis and **continue** applying the simplex method on the phase-II LP.

Example 1: Phase I

- Consider an LP

$$\begin{array}{rrcrcl} \max & x_1 & + & x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \geq & 6 \\ & x_1 & + & 2x_2 & \leq & 6 \\ & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2.} \end{array}$$

  which has no trivial bfs (due to the "≥" constraint).
- Its Phase-I standard form LP is

$$\begin{array}{rrcrcrcrcrcl} \min & & & & & & & & & x_5 & & \\ \text{s.t.} & 2x_1 & + & x_2 & - & x_3 & & & + & x_5 & = & 6 \\ & x_1 & + & 2x_2 & & & + & x_4 & & & = & 6 \\ & \multicolumn{11}{l}{x_i \geq 0 \quad \forall i = 1, ..., 5.} \end{array}$$

  - We need only one artificial variable x_5. x_3 and x_4 are slack variables.

Example 1: preparing the initial tableau

- Let's try to solve the Phase-I LP. First, let's prepare the initial tableau:

0	0	0	0	-1	0
2	1	-1	0	1	$x_5 = 6$
1	2	0	1	0	$x_4 = 6$

- Is this a valid tableau? No!
  - For all basic columns (in this case, columns 4 and 5), the 0th row should contain 0.
  - So we need to first **adjust the 0th row** with elementary row operations.

Example 1: preparing the initial tableau

- Let's adjust row 0 by adding row 1 to row 0.

0	0	0	0	−1	0
2	1	−1	0	1	$x_5 = 6$
1	2	0	1	0	$x_4 = 6$

$\overset{\text{adjust}}{\rightarrow}$

2	1	−1	0	0	6
2	1	−1	0	1	$x_5 = 6$
1	2	0	1	0	$x_4 = 6$

- Now we have a valid initial tableau to start from!
- The current bfs is $x^0 = (0, 0, 0, 6, 6)$, which corresponds to an **infeasible** solution to the original LP.
  - We know this because there are positive artificial variables.

Example 1: solving the Phase-I LP

- Solving the Phase-I LP takes only one iteration:

$$\begin{array}{ccccc|c} 2 & 1 & -1 & 0 & 0 & 6 \\ \hline \boxed{2} & 1 & -1 & 0 & 1 & x_5 = 6 \\ 1 & 2 & 0 & 1 & 0 & x_4 = 6 \end{array} \quad \rightarrow \quad \begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ \hline 1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & 1 & x_4 = 3 \end{array}$$

  - Whenever an artificial variable leaves the basis, we will not need to enter it again. Therefore, we may remove that column to save calculations.
- As we can remove all artificial variables, the original LP is feasible.
- A feasible basis for the original LP is $\{x_1, x_4\}$.

Example 1: solving the Phase-II LP

- Now let's construct the Phase-II LP.
- Step 1: put the original objective function "max $x_1 + x_2$" back:

$$
\begin{array}{cccc|c}
-1 & -1 & 0 & 0 & 0 \\
\hline
1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
0 & \frac{3}{2} & \frac{1}{2} & 1 & x_4 = 3
\end{array}
$$

- Is this a valid tableau? No!
  - Column 1, which should be basic, contains a nonzero number in the 0th row. It must be adjusted to 0.
- Before we run iterations, let's adjust the 0th row again.

Example 1: solving the Phase-II LP

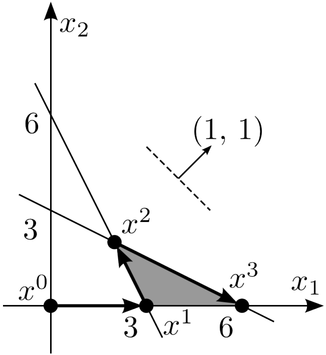
- Let's fix the 0th row and then run two iterations.

$$\begin{array}{cccc|c} -1 & -1 & 0 & 0 & 0 \\ \hline 1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & 1 & x_4 = 3 \end{array} \underbrace{\rightarrow}_{\text{adjust}} \begin{array}{cccc|c} 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 3 \\ \hline 1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\ 0 & \boxed{\frac{3}{2}} & \frac{1}{2} & 1 & x_4 = 3 \end{array}$$

$$\rightarrow \begin{array}{cccc|c} 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 4 \\ \hline 1 & 0 & -\frac{2}{3} & -\frac{1}{3} & x_1 = 2 \\ 0 & 1 & \boxed{\frac{1}{3}} & \frac{2}{3} & x_2 = 2 \end{array} \rightarrow \begin{array}{cccc|c} 0 & 1 & 0 & 1 & 6 \\ \hline 1 & 2 & 0 & 1 & x_1 = 6 \\ 0 & 3 & 1 & 2 & x_3 = 6 \end{array}$$

- The optimal bfs is $(6, 0, 6, 0)$.

Example 1: visualization



- x^0 is infeasible (the artificial variable x_5 is positive).
- x^1 is the initial bfs (as a result of Phase I).
- x^3 is the optimal bfs (as a result of Phase II).

Example 2: Phase-I LP

- Consider another LP

$$\begin{array}{rrcrcl} \max & x_1 & + & x_2 & & \\ \text{s.t.} & 2x_1 & + & x_2 & \geq & 6 \\ & x_1 & + & 2x_2 & = & 6 \\ & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2} \end{array}$$

  and its Phase-I LP

$$\begin{array}{rrcrcrcrcrcl} \min & & & & & & & x_4 & + & x_5 & & \\ \text{s.t.} & 2x_1 & + & x_2 & - & x_3 & + & x_4 & & & = & 6 \\ & x_1 & + & 2x_2 & & & & & + & x_5 & = & 6 \\ & \multicolumn{11}{l}{x_i \geq 0 \quad \forall i = 1, ..., 5.} \end{array}$$

- Please note that there are two artificial variables x_4 and x_5 (why?).
  - How about x_3?

Example 2: solving the Phase-I LP

- We first fix the 0th row and then run two iterations to remove all the artificial variables:

$$\begin{array}{ccccc|c} 0 & 0 & 0 & -1 & -1 & 0 \\ \hline 2 & 1 & -1 & 1 & 0 & x_4 = 6 \\ 1 & 2 & 0 & 0 & 1 & x_5 = 6 \end{array} \quad \overset{\text{adjust}}{\overbrace{\rightarrow}} \quad \begin{array}{ccccc|c} 3 & 3 & -1 & 0 & 0 & 12 \\ \hline \boxed{2} & 1 & -1 & 1 & 0 & x_4 = 6 \\ 1 & 2 & 0 & 0 & 1 & x_5 = 6 \end{array}$$

$x^0 = (0, 0, 0, \underline{6}, \underline{6})$ is infeasible

$$\rightarrow \quad \begin{array}{cccc|c} 0 & \frac{3}{2} & \frac{1}{2} & 0 & 3 \\ \hline 1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\ 0 & \boxed{\frac{3}{2}} & \frac{1}{2} & 1 & x_5 = 3 \end{array} \quad \rightarrow \quad \begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ \hline 1 & 0 & -\frac{2}{3} & x_1 = 2 \\ 0 & 1 & \frac{1}{3} & x_2 = 2 \end{array}$$

$x^1 = (3, 0, 0, \underline{0}, \underline{3})$ is infeasible

$x^2 = (2, 2, 0, \underline{0}, \underline{0})$ is feasible

Example 2: solving the Phase-II LP

- With the initial basis $\{x_1, x_2\}$, we then solve the Phase-II LP in one iteration (do not forget to fix the 0th row).[2]

$$\begin{array}{ccc|c} -1 & -1 & 0 & 0 \\ \hline 1 & 0 & -\frac{2}{3} & x_1 = 2 \\ 0 & 1 & \frac{1}{3} & x_2 = 2 \end{array} \quad \underbrace{\rightarrow}^{\text{adjust}} \quad \begin{array}{ccc|c} 0 & 0 & -\frac{1}{3} & 4 \\ \hline 1 & 0 & -\frac{2}{3} & x_1 = 2 \\ 0 & 1 & \boxed{\frac{1}{3}} & x_2 = 2 \end{array}$$

$x^2 = (2, 2, 0)$ is not optimal

$$\rightarrow \quad \begin{array}{ccc|c} 0 & 1 & 0 & 6 \\ \hline 1 & 2 & 0 & x_1 = 6 \\ 0 & 3 & 1 & x_3 = 6 \end{array}$$

$x^3 = (6, 0, 6)$ is optimal

[2] Would you visualize the whole process by yourself?

Example 3: Phase-I LP

- Consider the LP

$$\begin{array}{rrcrcr} \max & x_1 & & & & \\ \text{s.t.} & 2x_1 & + & x_2 & \leq & 4 \\ & x_1 & + & x_2 & = & 6 \\ & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2} \end{array}$$

and its Phase-I LP

$$\begin{array}{rrcrcrcrcr} \min & & & & & & & x_4 & & \\ \text{s.t.} & 2x_1 & + & x_2 & + & x_3 & & & = & 4 \\ & x_1 & + & x_2 & & & + & x_4 & = & 6 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.} \end{array}$$

Example 3: solving the Phase-I LP

- After adjusting the 0th row, we run two iterations:

$$\begin{array}{cccc|c} 0 & 0 & 0 & -1 & 0 \\ \hline 2 & 1 & 1 & 0 & x_3 = 4 \\ 1 & 1 & 0 & 1 & x_4 = 6 \end{array} \quad \overset{\text{adjust}}{\overbrace{\rightarrow}} \quad \begin{array}{cccc|c} 1 & 1 & 0 & 0 & 6 \\ \hline \boxed{2} & 1 & 1 & 0 & x_3 = 4 \\ 1 & 1 & 0 & 1 & x_4 = 6 \end{array}$$

$$x^0 = (0, 0, 4, \underline{6}) \text{ is infeasible}$$

$$\rightarrow \quad \begin{array}{cccc|c} 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 4 \\ \hline 1 & \boxed{\frac{1}{2}} & \frac{1}{2} & 0 & x_1 = 2 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 & x_4 = 4 \end{array} \quad \rightarrow \quad \begin{array}{cccc|c} -1 & 0 & -1 & 0 & 2 \\ \hline 2 & 1 & 1 & 0 & x_2 = 4 \\ -1 & 0 & -1 & 1 & x_4 = 2 \end{array}$$

$$x^1 = (0, 2, 0, \underline{4}) \text{ is infeasible} \qquad x^2 = (0, 4, 0, \underline{2}) \text{ is infeasible}$$

Example 3: solving the Phase-I LP

- The final tableau

-1	0	-1	0	2
2	1	1	0	$x_2 = 4$
-1	0	-1	1	$x_4 = 2$

  is optimal (for the Phase-I LP).
- However, in the Phase-I optimal solution $(0, 4, 0, 2)$, the artificial variable x_4 is still in the basis (and positive).
- Therefore, we conclude that the original LP is infeasible.[3]

[3] Try to visualize this!

Road map

- Information on tableaus.
- Finding an initial bfs.
- **Degeneracy and efficiency**.
- The matrix way of doing simplex.

Degeneracy

- Recall that an LP is **degenerate** if multiple bases correspond to a single basic solution.
- As an example, consider the following LP

$$\begin{array}{rrcrcl} \max & x_1 & + & 3x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 3 \\ & 2x_1 & + & 3x_2 & \leq & 6 \\ & \multicolumn{5}{l}{x_i \geq 0 \quad \forall i = 1, 2} \end{array}$$

and its standard form

$$\begin{array}{rrcrcrcrcl} \max & x_1 & + & 3x_2 & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & & & = & 3 \\ & 2x_1 & + & 3x_2 & & & + & x_4 & = & 6 \\ & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, ..., 4.} \end{array}$$

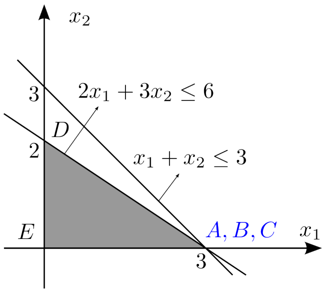
Degeneracy

- The six bases of

$$\begin{array}{rl} \max & x_1 + 3x_2 \\ \text{s.t.} & x_1 + x_2 + x_3 = 3 \\ & 2x_1 + 3x_2 + x_4 = 6 \\ & x_i \geq 0 \quad \forall i = 1, ..., 4 \end{array}$$

correspond to four distinct basic solutions.

Basis	Extreme point	Basic solution			
		x_1	x_2	x_3	x_4
$\{x_1, x_2\}$	A	3	0	0	0
$\{x_1, x_3\}$	B	3	0	0	0
$\{x_1, x_4\}$	C	3	0	0	0
$\{x_2, x_3\}$	D	0	2	1	0
$\{x_2, x_4\}$	–	0	3	0	-3
$\{x_3, x_4\}$	E	0	0	3	6



Impact of degeneracy

- In a degenerate LP, multiple feasible bases correspond to the same bfs.
- For the simplex method, it is possible to move to **another** basis but still at the **same** bfs.
- Running an iteration may have **no improvement**!
- Let's run the simplex method on this example.

Solving degenerate LPs

- After three iterations, we find an optimal solution:

$$\begin{array}{cccc|c} -1 & -3 & 0 & 0 & 0 \\ \hline \boxed{1} & 1 & 1 & 0 & x_3 = 3 \\ 2 & 3 & 0 & 1 & x_4 = 6 \end{array} \rightarrow \begin{array}{cccc|c} 0 & -2 & 1 & 0 & 3 \\ \hline 1 & 1 & 1 & 0 & x_1 = 3 \\ 0 & \boxed{1} & -2 & 1 & x_4 = 0 \end{array}$$

$$\rightarrow \begin{array}{cccc|c} 0 & 0 & -3 & 2 & 3 \\ \hline 1 & 0 & \boxed{3} & -1 & x_1 = 3 \\ 0 & 1 & -2 & 1 & x_2 = 0 \end{array} \rightarrow \begin{array}{cccc|c} 1 & 0 & 0 & 1 & 6 \\ \hline \frac{1}{3} & 0 & 1 & -\frac{1}{3} & x_3 = 1 \\ \frac{2}{3} & 1 & 0 & \frac{1}{3} & x_2 = 2 \end{array}$$

- In the second iteration, there is no improvement!
- The basis changes but the bfs does not change.

Efficiency of the simplex method

- In general, when we use the simplex method to solve a degenerate LP, there may be some iterations that have no improvements.
  - That may happen when multiple rows win the ratio test **at the same time**; those basic variables become 0 simultaneously.
- For some (very strange) instances, the simplex method needs to travel through **all** the bfs before it can make a conclusion.
- Therefore, the simplex method is an **exponential-time** algorithm.[4]
  - It may take an unacceptable long time to solve an LP.
- There are polynomial-time algorithms for Linear Programming.
  - For many practical problems, the simplex method is still faster.
- The simplex method is the most popular method for LP in industry.

[4]The number of iteration is $O(\binom{n}{m})$.

Efficiency of the simplex method

- When using the simplex method to solve an (original) LP, the number of **functional constraints** (m) greatly affects the computation time.
  - The computation time is roughly $O(m^3)$: proportional to the **cube** of the number of functional constraints.
  - Intuition: Number of iterations is $O(m)$ and number of operations in an iteration is $O(m^2)$.
- The number of variables, on the contrary, is not so important.
  - We calculate $x_B = A_B^{-1}b$ in each iteration, and $A_B \in \mathbb{R}^{m \times m}$.
- The **sparsity** of the coefficient matrix A is also important.
  - A is sparse means it has many zeros.
  - Practical problems typically have sparse coefficient matrices.
- For more information, see Chapters 5 and 7 (which will not be covered in this course).

Cycling

- One thing is even worse than running for a long time.
- At a degenerate bfs, the simplex method may enter an infinite loop! This is called **cycling**.
  - Basis 1 $\rightarrow$ basis 2 $\rightarrow$ basis 3 $\rightarrow \cdots \rightarrow$ basis 1.
- This may happen when we use a "not so good" way of selecting entering and leaving variables.
  - If we select the nonbasic variable with the "most significant reduced cost", cycling may occur.
- There are at least two ways to avoid cycling:
  - Randomize the selection of variables.
  - Apply an **anti-cycling** variable selection rule.

The smallest index rule

- One anti-cycling rule is the **smallest index rule**:[5]

Proposition 2 (The smallest index rule)

Using the following rule guarantees to solve a minimization LP in finite steps:

- *Among nonbasic variables with positive reduced costs, pick the one with the smallest index to enter the basis.*
- *Among basic variables that have the smallest valid ratios, pick the one with smallest index to exist.*

- The smallest index rule may not generate the **least iterations** toward an optimal solution.
  - No variable selection rule can guarantee to be the most efficient!
- The smallest index rule can guarantee **no cycling**!

[5] Developed by Bland in 1977.

Road map

- Information on tableaus.
- Finding an initial bfs.
- Degeneracy and efficiency.
- **The matrix way of doing simplex**.

Implementation of the simplex method

- When one implements the simplex method with computer programs, using tableaus is not the most efficient way.
- Using **matrices** is the most efficient.
- Recall that the standard form LP can be expressed as

$$\begin{array}{rl} \min & c_B^T A_B^{-1} b - \left(c_B^T A_B^{-1} A_N - c_N^T\right) x_N \\ \text{s.t.} & x_B = A_B^{-1} b - A_B^{-1} A_N x_N \\ & x_B, x_N \geq 0 \end{array}$$

or

$$\begin{array}{rcrcl} z & + & \left(c_B^T A_B^{-1} A_N - c_N^T\right) x_N & = & c_B^T A_B^{-1} b \\ I x_B & + & A_B^{-1} A_N x_N & = & A_B^{-1} b. \end{array}$$

- We may do **matrix operations** to do iterations.

At any feasible basis

$$\begin{aligned} z \quad &+ \quad (c_B^T A_B^{-1} A_N - c_N) x_N &= \quad c_B^T A_B^{-1} b \\ I x_B \quad &+ \quad A_B^{-1} A_N x_N &= \quad A_B^{-1} b. \end{aligned}$$

- At any feasible basis B:
  - The current bfs is $x = (x_B, x_N) = (A_B^{-1}b, 0)$ and the current $z = c_B^T A_B^{-1} b$.
- For the entering variable:
  - The **reduced costs** are $\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T$.
  - The reduced cost of variable x_j is $\bar{c}_j = c_B^T A_B^{-1} A_j - c_j$ for all $j \in N$.
  - If there exists $j \in N$ such that $\bar{c}_j > 0$, x_j may enter.
- For the leaving variable:
  - If x_j enters, the **ratio test** is to compare the ratios $\dfrac{(A_B^{-1}b)_i}{(A_B^{-1}A_j)_i}$.
  - The basic variable corresponding to row i may leave if $(A_B^{-1}A_j)_i > 0$ and

$$\frac{(A_B^{-1}b)_i}{(A_B^{-1}A_j)_i} \leq \frac{(A_B^{-1}b)_k}{(A_B^{-1}A_j)_k} \quad \forall k = 1, ..., m \text{ such that } (A_B^{-1}A_j)_k > 0.$$

When we stop

- At any optimal basis B, we know that
  - The reduced costs $\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T \leq 0$.
  - The optimal bfs is $x^* = (x_B^*, x_N^*) = (A_B^{-1} b, 0)$.
  - The current objective value is $z^* = c_B^T A_B^{-1} b$.
- To detect multiple optimal solutions:
  - $\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T \leq 0$.
  - There exists $j \in N$ such that $\bar{c}_j = 0$.
- To detect unboundedness:
  - There exists $j \in N$ such that $\bar{c}_j > 0$.
  - Moreover, $(A_B^{-1} A_j)_i \leq 0$ for all $i \in B$.

Example

- Consider the example again:

$$\begin{array}{llllllllllll} \min & -x_1 & & & & & & & & & & \\ \text{s.t.} & 2x_1 & - & x_2 & + & x_3 & & & & & = & 4 \\ & 2x_1 & + & x_2 & & & + & x_4 & & & = & 8 \\ & & & x_2 & & & & & + & x_5 & = & 3 \\ & & x_i \geq 0 & \forall i = 1, ..., 5. & & & & & & & & \end{array}$$

- In the matrix representation, we have

$$c^T = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and } b = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix}.$$

A feasible basis

- Given $x_B = (x_1, x_4, x_5)$ and $x_N = (x_2, x_3)$, we have

$$c_B^T = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, \quad c_N^T = \begin{bmatrix} 0 & 0 \end{bmatrix},$$
$$A_B = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix}.$$

- Given the basis, we have

$$x_B = A_B^{-1} b = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_4 \\ x_5 \end{bmatrix}, \quad \text{and}$$
$$z = c_B^T A_B^{-1} b = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = -2.$$

- The current bfs is $x = (x_1, x_2, x_3, x_4, x_5) = (2, 0, 0, 4, 3)$.

A feasible basis

- For $x_N = (x_2, x_3)$, the reduced costs are

$$\begin{aligned}\bar{c}_N^T &= c_B^T A_B^{-1} A_N - c_N^T \\ &= \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.\end{aligned}$$

- x_2 enters. For $x_B = (x_1, x_4, x_5)$, we have
  - $A_B^{-1} A_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \end{bmatrix}$ and $A_B^{-1} b = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.
  - $\frac{4}{2} < \frac{3}{1}$, so x_4 leaves.

An optimal basis

- Given $x_B = (x_1, x_2, x_5)$ and $x_N = (x_3, x_4)$ we have

$$c_B^T = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, \quad c_N^T = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$A_B = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix}.$$

- Given the basis, we have

$$x_B = A_B^{-1} b = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix}, \quad \text{and}$$

$$z = c_B^T A_B^{-1} b = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = -3.$$

- The current bfs is $x = (x_1, x_2, x_3, x_4, x_5) = (3, 2, 0, 0, 1)$.

An optimal basis

- For $x_N = (x_3, x_4)$, the reduced costs are

$$
\begin{aligned}
\bar{c}_N^T &= c_B^T A_B^{-1} A_N - c_N^T \\
&= \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}.
\end{aligned}
$$

- No variable should enter: This bfs is optimal.

The matrix way

- In short, the simplex method may be run with matrix calculations.
- In this way, the bottleneck is the calculation of A_B^{-1}.
- Nevertheless, because the current basis B and the previous one have only **one variable** different, the current A_B and the previous one have only **one column** different.
  - Calculating A_B^{-1} can be faster with the previous one.[6]
- In fact, how do you know that A_B is still **invertible** after changing one column?

[6] Section 5.4 contains relevant discussion about calculating A_B^{-1}.