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Game Theory (Part 1): Static Games

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Brief history of game theory

- So far we have focused on decision making problems with only one decision maker.
- **Game theory** provides a framework for analyzing **multi-player** decision making problems.
- While it has been implicitly discussed in Economics for more than 200 years, game theory is established as a field in 1934.
  - In 1934, John von Neumann and Oskar Morgenstern published a book *Theory of games and economic behaviors*.
- Since then, game theory has been widely studied, applied, and discussed in mathematics, economics, operations research, industrial engineering, computer science, etc.
  - Actually almost all fields of social sciences and business have game theory involved in.
  - The Nobel Prizes in economic sciences have been honored to game theorists (broadly defined) in 1994, 1996, 2001, 2005, 2007, and 2012.

Road map

- **Introduction**.
- Nash equilibrium.
- Retailer competitions.

Prisoners' dilemma: story

- A and B broke into a grocery store and stole some money. Before police officers caught them, they hided those money carefully without leaving any evidence. However, a monitor got their images when they broke the window.
- They were kept in two separated rooms. Each of them were offered two choices: **denial or confession**.
  - If both of them deny the fact of stealing money, they will both get one month in prison.
  - If one of them confesses while the other one denies, the former will be set free while the latter will get nine months in prison.
  - If both confesses, they will both get six months in prison.
- They **cannot communicate** and must make choices **simultaneously**.
- What will they do?

Prisoners' dilemma: formulation

- We may use the following matrix to summarize this "**game**":

Player 2

		Denial	Confession
Player 1	Denial	$-1, -1$	$-9, 0$
	Confession	$0, -9$	$-6, -6$

  - There are two **players**. Player 1 is the **row player** and player 2 is the **column player**.
  - For each combination of actions, the two numbers are the **payoffs** under their actions: the first for player 1 and the second for player 2.
  - E.g., if both prisoners deny, they will both get one month in prison, which is represented by a payoff of -1.
  - E.g., if prisoner 1 denies and prisoner 2 confesses, prisoner 1 will get 0 month in prison (and thus a payoff 0) and prisoner 2 will get 9 months in prison (and thus a payoff -9).

Prisoners' dilemma: solution

- Let's **solve** this game by **predicting** what they will/may do.

Player 2

		Denial	Confession
Player 1	Denial	$-1, -1$	$-9, 0$
	Confession	$0, -9$	$-6, -6$

- Player 1 thinks:
  - "If he denies, I should confess."
  - "If he confesses, I should still confess."
  - "I see! I should **confess** anyway!"
- For player 2, the situation is the same and he will also **confess**.
- The **solution** of this game, i.e., the **equilibrium outcome**, is that both prisoner confess.
- Note that this outcome can be "improved" if they **cooperate**.
  - This situation is said to be (socially) **inefficient**.

Static games

- A game like the prisoners' dilemma in which all players choose their actions **simultaneously** is called a **static game**.
- This question (with a different story) was first raised by Professor Tucker (one of the names in the KKT condition) in a seminar.
- In this game, confession is said to be a **dominant strategy**.
  - A dominant strategy should be chosen anyway.
- **Lack of coordination** can result in a **lose-lose** outcome.
- Interestingly, even if they have promised each other to deny once they are caught, this promise is **non-credible**. Both of them will still confess to maximize their payoffs.

Applications of prisoners' dilemma

- Two companies are both active in a market. At this moment, they both earn \$4 million dollars per year.
- Each of them may advertise with an annual cost of \$3 million:
  - If one advertises while the other does not, she earns \$9 millions and the competitor earns \$1 million.
  - If both advertise, both will earn \$6 millions.

	Advertise	Be silent
Advertise	$3, 3$	$6, 1$
Be silent	$1, 6$	$4, 4$

- What will they do?

- Two countries are neighbors.
- Each of them may choose to develop a new weapon:
  - If one does so while the other one keeps the current status, the former's payoff is 20 and the latter's payoff is -100.
  - If both do this, however, their payoffs are both -10.

	MW	CS
MW	$-10, -10$	$20, -100$
CS	$-100, 20$	$0, 0$

- What will they do?

Predicting the outcome of other games

- How about games that are not a prisoners' dilemma? Do we have a systematic way to predict the outcome?
- What will be the outcome (a combination of actions chosen by the two players) of the following game?

	Left	Middle	Right
Up	1, 0	1, 2	0, 1
Down	0, 3	0, 1	2, 0

Eliminating strictly dominated options

- We may apply the same trick we used to solve the prisoners' dilemma.
- For player 2, playing Middle **strictly dominates** playing Right. So we may **eliminate** the column of Right without eliminating any possible outcome:

	Left	Middle	Right
Up	1, 0	1, 2	0, 1
Down	0, 3	0, 1	2, 0

→

	Left	Middle
Up	1, 0	1, 2
Down	0, 3	0, 1

- Now, player 1 knows that player 2 will never play Right. Down is thus dominated by Up and can be eliminated.

	Left	Middle
Up	1, 0	1, 2
Down	0, 3	0, 1

→

	Left	Middle
Up	1, 0	1, 2

- What is the outcome of this game?

Eliminating strictly dominated options

- In game theory, options are typically called **strategies**.
- The above idea is called the **iterative elimination** of **strictly dominated strategies**.
- It solves some games. However, is also fails to solve some others.
- Consider the following game "Matching pennies":

	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

- What may we do when no strategies can be eliminated?
- In 1950, John Nash developed the concept of **equilibrium solutions**, which are called **Nash equilibria** nowadays.

Road map

- Introduction.
- **Nash equilibrium**.
- Retailer competitions.

Nash equilibrium: definition

- The concept of Nash equilibrium is defined as follows:

Definition 1

For an n-player game, let S_i be player i's action space and u_i be player i's utility function, $i = 1, ..., n$. An action profile $(s_1^*, ..., s_n^*)$, $s_i^* \in S_i$, is a Nash equilibrium if

$$\begin{aligned} & u_i(s_1^*, ..., s_{i-1}^*, s_i^*, s_{i+1}^*, ..., s_n^*) \\ \geq\ & u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*) \end{aligned}$$

for all $s_i \in S_i$, $i = 1, ..., n$.

- In other words, s_i^* is optimal to $\max\limits_{s_i \in S_i} u_i(s_1^*, ..., s_{i-1}^*, s_i, s_{i+1}^*, ..., s_n^*)$.
- If all players are choosing a strategy in a Nash equilibrium, no one has an incentive to **unilaterally deviates**.

Nash equilibrium: an example

- Consider the following game in which no action is strictly dominated:

	L	C	R
T	$0, 7$	$7, 0$	$5, 4$
M	$7, 0$	$0, 7$	$5, 4$
B	$4, 5$	$4, 5$	$6, 6$

- What is a Nash equilibrium?
  - (T, L) is not: Player 1 will unilaterally deviate to M or B.
  - (T, C) is not: Player 2 will unilaterally deviate to L or R.
  - (B, R) is: No one will unilaterally deviate.
  - Any other Nash equilibrium?

Nash equilibrium as a solution concept

- In a static game, a Nash equilibrium is a reasonable outcome.
  - Imagine that the players play this game **repeatedly**.
  - If they happen to be in a Nash equilibrium, no one has the incentive to **unilaterally deviate**, i.e., to change her action while all others keep their actions.
  - If they do not, at least one will deviate. This process will continue until a Nash equilibrium is reached.
- For example, if they starts at (T, L), eventually they will stop at (B, R), the unique Nash equilibrium of this game.

	L	C	R
T	$0, 7$	$7, 0$	$5, 4$
M	$7, 0$	$0, 7$	$5, 4$
B	$4, 5$	$4, 5$	$6, 6$

- A non-Nash solution is **unstable**.

Nash equilibrium: More examples

- Is there any Nash equilibrium of the prisoners' dilemma?

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

- Is there any Nash equilibrium of the game "BoS"?
  - Battle of sexes.
  - Bach or Stravinsky.

	Denial	Confession
Denial	$-1, -1$	$-9, 0$
Confession	$0, -9$	$-6, -6$

- Is there any Nash equilibrium of the matching pennies game?

	Denial	Confession
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

Road map

- Introduction.
- Nash equilibrium.
- **Retailer competitions**.

Cournot Competition

- In 1838, Antoine Cournot introduced the following **quantity competition** of a homogeneous product between two retailers.
- Let q_i be the production quantity of firm i, $i = 1, 2$.
- The market-clearing price p of the product depends on the aggregate demand $q = q_1 + q_2$:

$$p = a - q = a - q_1 - q_2.$$

- Unit production cost of both firms is $c < a$.
- Our questions are:
  - In this environment, what will these two firms do?
  - Is the outcome efficient?
  - What is the difference between monopoly and duopoly (i.e., integration and decentralization).

Formulations

- Suppose they cooperate (collude) in making this decision:

$$\pi^{\mathrm{C}} = \max_{q_1 \geq 0, q_2 \geq 0} \; q_1(a - q_1 - q_2 - c) + q_2(a - q_1 - q_2 - c).$$

  - The unique optimal solution is $q_1^{**} = q_2^{**} = \frac{a-c}{4}$ with $\pi^{\mathrm{C}} = \frac{(a-c)^2}{4}$.
- Suppose two firms are making their decisions:
  - Firm 1 and firm 2 simultaneously solve their problems

$$\pi_1^{\mathrm{D}} = \max_{q_1 \geq 0} \; u_1(q_1 | q_2) \quad \text{and} \quad \pi_2^{\mathrm{D}} = \max_{q_2 \geq 0} \; u_2(q_2 | q_1),$$

    where their payoff functions are

$$u_i(q_i | q_{3-i}) = q_i(a - q_i - q_{3-i} - c) \quad \forall i = 1, 2.$$

  - As for an outcome, we look for a Nash equilibrium.

Formulations

- If (q_1^*, q_2^*) is a Nash equilibrium, it must leave no incentive for either firm to unilaterally deviate.
  - For firm 1, that means q_1^* is **optimal** given that firm 2 chooses q_2^*.
  - In this case, firm 1's problem is

$$\max_{q_1 \geq 0} u_1(q_1|q_2^*) = \max_{q_1 \geq 0} q_1(a - q_1 - q_2^* - c)$$

  - The FOC requires

$$u_1'(q_1|q_2^*)|_{q_1=q_1^*} = a - 2q_1^* - q_2^* - c = 0,$$

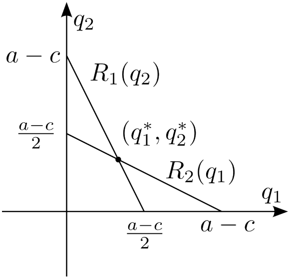
    i.e., $q_1^* = \frac{1}{2}(a - q_2 - c)$ (is it optimal?).
  - In fact, $R_1(q_2) = \frac{1}{2}(a - q_2 - c)$ is firm 1's **best response function** given any firm 2's action q_2.
- Similarly, for firm 2 we need $q_2^* = \frac{1}{2}(a - q_1^* - c)$.
  - Firm 2's best response to firm 1's action q_1 is $R_2(q_1) = \frac{1}{2}(a - q_1 - c)$.

Solving the Cournot competition

- Let's use the two equalities:
  - If (q_1^*, q_2^*) is a Nash equilibrium, it must satisfy

$$q_1^* = \frac{1}{2}(a - q_2^* - c) \quad \text{and} \quad q_2^* = \frac{1}{2}(a - q_1^* - c).$$

  - The unique solution to this system is $q_1^* = q_2^* = \frac{a-c}{3}$.
- Or we may use the two best response functions:
  - A Nash equilibrium always lies on an **intersection** of all the best response functions.



- **In equilibrium**, firm i earns

$$\pi_i^{\mathrm{D}} = \frac{(a-c)}{3}\left[a - \frac{2(a-c)}{3} - c\right] = \frac{(a-c)^2}{9}.$$

Distortion due to decentralization

- Comparison:

Scenario	Aggregate quantity	Aggregate profit
Integration	$q^{**} = \frac{a-c}{2}$	$\pi^{\text{C}} = \frac{(a-c)^2}{4}$
Decentralization	$q_1^* + q_2^* = \frac{2(a-c)}{3}$	$\pi_1^{\text{D}} + \pi_2^{\text{D}} = \frac{2(a-c)^2}{9}$

- For profits, integration results in **win-win** and is more efficient.
- For quantities:
  - If they cooperate, each will order $\frac{a-c}{4}$.
  - Once they do not cooperate, each will order $\frac{a-c}{3}$.
  - Why does one intend to **increase** its quantity under decentralization?
- $(q_1, q_2) = (\frac{a-c}{4}, \frac{a-c}{4})$ profit-improving but **not** a Nash equilibrium:
  - If $q_2' = \frac{a-c}{4}$, firm 1 deviates to $q_1'' = R_1(q_2') = \frac{1}{2}(a - q_2' - c) = \frac{3(a-c)}{8}$.
  - This a **prisoners' dilemma**!

Inefficiency due to decentralization

- How about consumers?
  - Under decentralization, the aggregate quantity is $\frac{2(a-c)}{3}$ and the market-clearing price is $\frac{a-c}{3}$.
  - Under integration, the aggregate quantity is $\frac{a-c}{2}$ and the market-clearing price is $\frac{a-c}{2}$.
- Under decentralization, **more** consumers buy this product with a **lower** price.
  - Consumers **benefits from competition**.
  - Integration benefits the firms but hurts consumers.

Bertrand competition

- In 1883, Joseph Bertrand considered another format of retailer competition: They choose **prices** instead of quantities.
- Firm i chooses price p_i, $i = 1, 2$.
- Firm i's demand quantity is

$$q_i = a - p_i + bp_{3-i}, i = 1, 2.$$

  - $b \in [0, 1)$ measures the **intensity of competition**: The larger b, the more intense the competition.
  - Why $b < 1$?
- Unit production cost $c < a$.

Solving the Bertrand competition

- Suppose (p_1^*, p_2^*) is a Nash equilibrium.
- For firm 1, p_1^* must be optimal to

$$\max_{p_1 \geq 0} \pi_1\big(p_1 | p_2^*\big) = \big(a - p_1 + bp_2^*\big)(p_1 - c).$$

  Therefore, $p_1^* = \frac{1}{2}(a + bp_2^* + c)$.
- Similarly, $p_2^* = \frac{1}{2}(a + bp_1^* + c)$.
- The unique Nash equilibrium is $p_1^* = p_2^* = \frac{a+c}{2-b}$.
- If they cooperate (collude), they solve

$$\max_{p_1 \geq 0, p_2 \geq 0} (a - p_1 + bp_2)(p_1 - c) + (a - p_2 + bp_1)(p_2 - c).$$

  - The unique optimal solution is $p_1^{**} = p_2^{**} = \frac{a+c(1-b)}{2(1-b)} > p_1^* = p_2^*$ (why?).
  - Why firms intend to decrease the price under decentralization?
  - Does integration hurt or benefit the firms? How about consumers?