# Operations Research, Spring 2016
# Homework 2

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## 1 Problems

1. (20 points) Consider the following integer program:

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 10 \\ & 3x_1 + 4x_2 \geq 25 \\ & x_i \in \mathbb{Z}_+ \quad \forall i = 1, 2 \end{aligned}$$

The set $\mathbb{Z}$ is the set of all integers and the set $\mathbb{Z}_+$ is the set of all nonnegative integers. The notation "$x_i \in \mathbb{Z}_+$" therefore means "$x_i$ is an integer and $x_i \geq 0$".

(a) (10 points) Use branch-and-bound to solve the integer program. Write down the complete branch-and-bound tree and an optimal solution.

(b) (5 points) For the linear relaxation of the IP, find its dual program.

(c) (5 points) For the linear relaxation of the IP, find the shadow prices of the two constraints.

2. (30 points) Ikuta is traveling from Japan to Taiwan and have bought a bag that can carry up to 20 kg of furniture. The weight and degree of importance of each item that she is considering carrying are given in the table below. Ikuta wants to maximize the total degree of importance of the items she carry while satisfying the capacity constraint.

| Item | Importance | Weight (kg) |
|---|---|---|
| 1 | 7 | 8 |
| 2 | 4 | 6 |
| 3 | 1 | 3 |
| 4 | 3 | 4 |
| 5 | 1 | 2 |
| 6 | 2 | 7 |
| 7 | 5 | 7 |

(a) (5 points) Formulate an integer program that solves her problem.

(b) (10 points) Apply branch-and-bound to solve Ikuta's problem. Write down the complete branch-and-bound tree and an optimal solution.

(c) (5 points) Find the dual LP of the linear relaxation of the IP in Part (a).

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(d) (5 points) Apply complementary slackness to find which dual constraints must be binding at a dual optimal solution.

(e) (5 points) For the linear relaxation of the IP in Part (a), find the shadow price of the capacity constraint.

**Hint.** You may, but you do not need to solve the primal LP by the simplex method or solve the dual LP. Use the definition of shadow prices!

3. (25 points) Ikuta is trying to choose seven out of ten volleyball players for a contest. Ten players have been rated (on a scale of 1 for poor to 5 for excellent) according to their spiking, blocking, receiving, and passing abilities. The positions that each player is allowed to play and the player's abilities are listed in the table below, where S, L, H, M, and O stand for setters, liberals, outside hitters, middle hitters, and opposite.

| Player | Position | Spiking | Blocking | Receiving | Passing |
|---|---|---|---|---|---|
| 1 | S | 2 | 1 | 3 | 5 |
| 2 | S or L | 2 | 3 | 4 | 3 |
| 3 | S or H or O | 4 | 4 | 4 | 3 |
| 4 | H or L or O | 3 | 3 | 5 | 3 |
| 5 | H or M | 5 | 4 | 3 | 3 |
| 6 | H or M or O | 3 | 3 | 3 | 2 |
| 7 | M | 4 | 5 | 1 | 1 |
| 8 | H or L | 4 | 2 | 5 | 2 |
| 9 | L | 1 | 1 | 5 | 3 |
| 10 | M or H | 2 | 3 | 3 | 2 |

The seven-player team must satisfy the following restrictions:

- There must be exactly one player playing setter, one playing liberal, two playing outside hitter, two playing middle hitter, and one playing opposite. For example, you may not select players 3, 4, 5, 6, 7, 8, and 10, because then only player 3 may play setter and opposite, but she may only play one position.
- To make the team flexible enough, there must be at least two players who may play setter and at least three players who may play outside hitter.
- If player 1 is chosen, then player 3 cannot be chosen.
- At least two among players 1, 3, 4, 6, 8 must be chosen.

Given these constraints, Ikuta wants to maximize

$$\text{total spiking level of the four hitters and opposite} + 3 \times \text{passing level of the setter}$$

of the starting team.

(a) Formulate an integer program that solves Ikuta's problem.

(b) Add constraints to the IP to account for the following restrictions:

- The average receiving level of the two outside hitters, opposite, and liberal must be at least 3.5.
- The average blocking level of the two middle hitters and the opposite must be at least 3.5.

4. (10 points) A city is divided into eight districts. The time (in minutes) it takes an ambulance to travel from one district to another is shown in the table below. The population of each district (in thousands) is as follows: district 1, 40; district 2, 30; district 3, 35; district 4, 20; district 5, 15; district 6, 50; district 7, 45; district 8, 60.

| District | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 3 | 4 | 6 | 8 | 9 | 8 | 10 |
| 2 | 3 | 0 | 5 | 4 | 8 | 6 | 12 | 9 |
| 3 | 4 | 5 | 0 | 2 | 2 | 3 | 5 | 7 |
| 4 | 6 | 4 | 2 | 0 | 3 | 2 | 5 | 4 |
| 5 | 8 | 8 | 2 | 3 | 0 | 2 | 2 | 4 |
| 6 | 9 | 6 | 3 | 2 | 2 | 0 | 3 | 2 |
| 7 | 8 | 12 | 5 | 5 | 2 | 3 | 0 | 2 |
| 8 | 10 | 9 | 7 | 4 | 4 | 2 | 2 | 0 |

The city has two ambulances and wants to locate them to maximize the number of people who live within 2 minutes of an ambulance. Formulate an integer program that can achieve this goal.

5. (15 points) Write AMPL to solve the following problems. For each problem, copy and paste all your AMPL programs to your answer sheet. Then write down an optimal solution found by AMPL (or conclude that the problem is infeasible or unbounded).

   (a) (5 points) Problem 1a

   (b) (10 points) Problem 2a.

## 2 Submission rules

The deadline of this homework is 2 pm, May 2, 2016. Please put a hard copy of the work into the instructor's mailbox on the first floor of the Management Building 2 by the due time. Works submitted between 2 pm and 3 pm will get 10 points deducted as a penalty. Submissions later than 3 pm will not be accepted. Each student must submit her/his individual work.