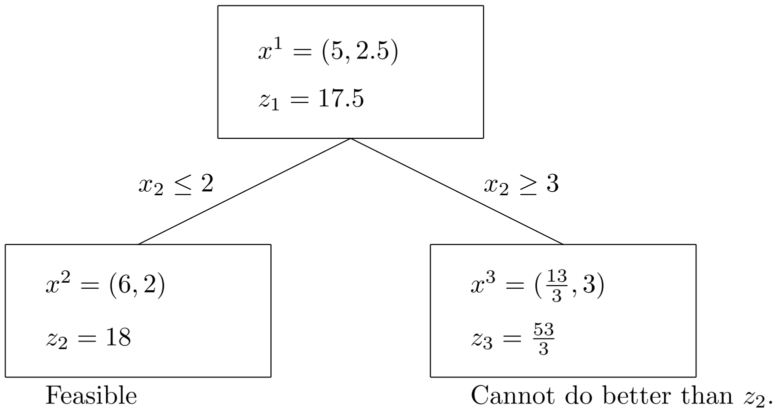
Operations Research, Spring 2016
Suggested Solution for Homework 2

Solution providers: Kiwi Liu and Johnny Chen
Department of Information Management
National Taiwan University

1. (a) The optimal solution is $x^2 = (6, 2)$ with the objective value $z^* = 18$.



(b)

$$\begin{array}{rl} \max & 10y_1 + 25y_2 \\ \text{s.t.} & y_1 + 3y_2 \leq 2 \\ & 2y_1 + 4y_2 \leq 3 \\ & y_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

(c) Since the shadow prices equal the values of dual variables in the dual optimal solution, we solve the dual program and get the dual optimal solution $(0.5, 0.5)$, which is the shadow prices.

2. (a) Let the parameters be

$$\begin{aligned} I_i &= \text{the importance of item } i,\ i = 1, ..., 7, \\ W_i &= \text{the weight of item } i,\ i = 1, ..., 7. \end{aligned}$$

Let the decision variables be

$$x_i = \begin{cases} 1 & \text{if Ikuta carries item } i \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., 7.$$

The integer program is

$$\begin{array}{rl} \max & \sum_{i=1}^{7} I_i x_i \\ \text{s.t.} & \sum_{i=1}^{7} W_i x_i \leq 20 \\ & x_i \in \{0, 1\} \quad \forall i = 1, ..., 7. \end{array}$$

(b) The branch-and-bound is shown in Figure 1. Then, we obtain the optimal solution $x_6 = (1, 1, 0, 1, 1, 0, 0)$ with objective value $z^* = 15$.

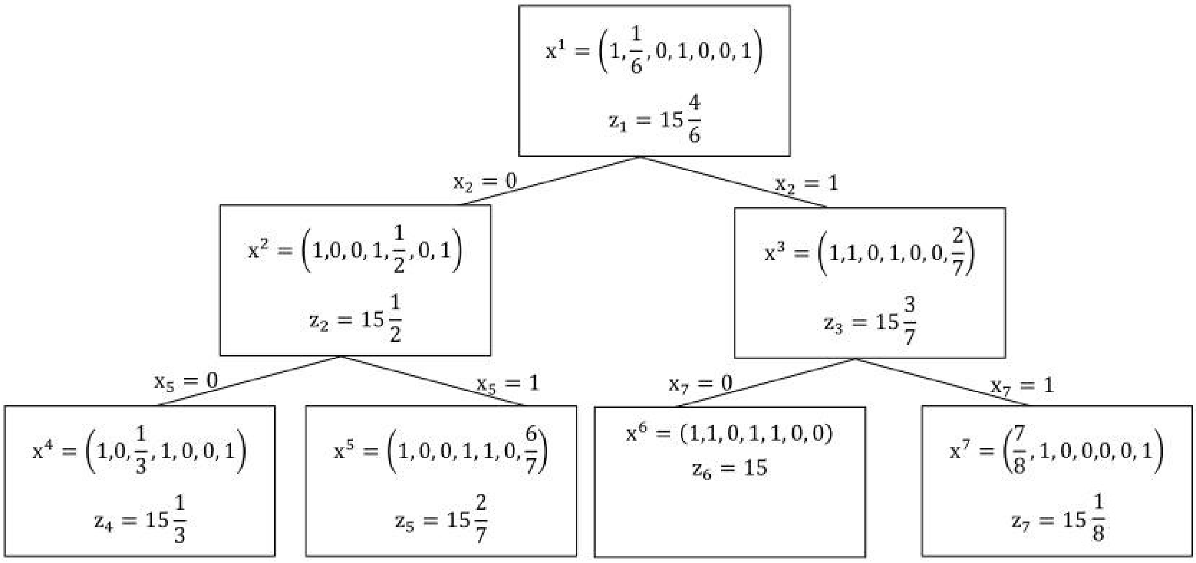


Figure 1: Branch-and-bound for Problem 2b

(c) The linear relaxation of the IP in Part (a) is

$$\begin{aligned} \max \quad & \sum_{i=1}^{7} I_i x_i \\ \text{s.t.} \quad & \sum_{i=1}^{7} W_i x_i \leq 20 \\ & 0 \leq x_i \leq 1 \quad \forall i = 1, ..., 7. \end{aligned}$$

The dual LP is

$$\begin{aligned} \min \quad & 20y + \sum_{i=1}^{7} u_i \\ \text{s.t.} \quad & W_i y + u_i \geq I_i \quad \forall i = 1, ..., 7 \\ & u_i \geq 0 \quad \forall i = 1, ..., 7 \\ & y \geq 0. \end{aligned}$$

(d) As we know, if a dual constraint is nonbinding, the corresponding primal variable is zero. By Part(b), we know that x_3, x_5, x_6 equals to zero, so the following dual constraints in Part(c) must be binding:

$$\begin{aligned} 8y + u_1 &\geq 7 \\ 6y + u_2 &\geq 4 \\ 4y + u_4 &\geq 3 \\ 7y + u_7 &\geq 5. \end{aligned}$$

(e) The shadow price of capacity constraint is the amount of objective value increased when we change 20 to 21 (the RHS is increased by 1). In this problem, we put the item into bag according to the ratio $\frac{\text{Importance}}{\text{Weight}}$. We then obtain the order $x_1, x_7, x_4, x_2, x_5, x_3, x_6$. We finds that no matter the capacity is 20 or 21, we will both stop when we put x_2 into bag. As the result, the shadow price is exactly the ratio of x_2, which is $\frac{2}{3}$.

3. We number ability 1-4 Spiking, Blocking, Receiving, Passing.
   Let the parameters be

$$P = \text{set of positions} = \{\text{S, L, H, M, O}\},$$

$$A_{ij} = \begin{cases} 1 & \text{if the player } i \text{ can play position } j \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., 10,\ j \in P,$$

$$R_{ij} = \text{the rating of ability } j \text{ of player } i,\ i = 1, ..., 10,\ j = 1, ..., 4.$$

   Let the decision variables be

$$x_{ij} = \begin{cases} 1 & \text{if the player } i \text{ is chosen to play position } j \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., 10,\ j \in P.$$

   (a)

$$\begin{aligned}
\max \quad & \sum_{i=1}^{10} (R_{i1}(x_{iH}A_{iH} + x_{iM}A_{iM} + x_{iO}A_{iO}) + 3R_{i4}x_{iS}A_{iS}) \\
\text{s.t.} \quad & \sum_{i=1}^{10} x_{ij}A_{ij} = 1 \quad \forall j = S, L, O \\
& \sum_{i=1}^{10} x_{ij}A_{ij} = 2 \quad \forall j = H, M \\
& \sum_{i=1}^{10} \sum_{j \in P} x_{ij}A_{iS} \geq 2 \\
& \sum_{i=1}^{10} \sum_{j \in P} x_{ij}A_{iH} \geq 3 \\
& \sum_{j \in P} (x_{1j} + x_{3j}) \leq 1 \\
& \sum_{j \in P} (x_{1j} + x_{3j} + x_{4j} + x_{6j} + x_{8j}) \geq 2 \\
& \sum_{j \in P} x_{ij} \leq 1 \quad \forall i = 1, ..., 10 \\
& x_{ij} \in \{0, 1\} \quad \forall i = 1, ..., 10,\ j \in P.
\end{aligned}$$

   (b) We add the following two constraints into the model in (a):

$$\frac{1}{4} \sum_{i=1}^{10} (R_{i3}(x_{iL}A_{iL} + x_{iH}A_{iH} + x_{iO}A_{iO})) \geq 3.5$$

$$\frac{1}{3} \sum_{i=1}^{10} (R_{i2}(x_{iM}A_{iM} + x_{iO}A_{iO})) \geq 3.5$$

4. Let the parameters be

$$P_i = \text{the population of district } i \text{ (in thousands)},\ i\text{=1,...8},$$

$$T_{ij} = \begin{cases} 1 & \text{if the time it takes an ambulance to travel from district } i \\ & \text{to district } j \text{ is less than 2 minutes,} \\ 0 & \text{otherwise} \end{cases}, i = 1, ...8,\ j = 1, ...8.$$

   Let the decision variables be

$$x_i = \begin{cases} 1 & \text{if the ambulance is located in district } i, \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., 8,$$

$$y_i = \begin{cases} 1 & \text{if there is an ambulance can arrive at the district } i \text{ within two minutes,} \\ 0 & \text{otherwise} \end{cases}, i = 1, ..., 8.$$

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{8} P_i y_i \\
\text{s.t.} \quad & \sum_{i=1}^{8} x_i = 2 \\
& \sum_{j=1}^{8} x_j T_{ij} \geq y_i \quad \forall i = 1, ..., 8 \\
& x_i,\ y_i \in \{0, 1\} \quad \forall i = 1, ..., 8.
\end{aligned}
$$

5. omitted