

Operations Research

The Simplex Method (Part 2)

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Introduction

- ▶ Last time we introduced the simplex method.
- ▶ There remain some unsolved problem:
 - ▶ How to find an initial bfs? How to know whether an LP is infeasible?
 - ▶ What if an LP is unbounded?
 - ▶ What if multiple nonbasic variables may be entered?
 - ▶ What if there is a tie in a ratio test?
 - ▶ How efficient the simplex method is?
- ▶ In this lecture, we will address these issues (and some more).
- ▶ Read Sections 4.5 and 4.6 thoroughly.
 - ▶ Sections 4.8 and 4.9 contain discussions regarding efficiency.

Road map

- ▶ **Information on tableaus.**
- ▶ Finding an initial bfs.
- ▶ Degeneracy and efficiency.
- ▶ The matrix way of doing simplex.

Identifying unboundedness

- ▶ When is an LP **unbounded**?
- ▶ An LP is unbounded if:
 - ▶ There is an improving direction.
 - ▶ Along that direction, we may move forever.
- ▶ When we run the simplex method, this can be easily checked in a simplex tableau.
- ▶ Consider the following example:

$$\begin{array}{ll} \max & x_1 \\ \text{s.t.} & x_1 - x_2 \leq 1 \\ & 2x_1 - x_2 \leq 4 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

Unbounded LPs

- The standard form is:

$$\begin{aligned}
 \max \quad & x_1 \\
 \text{s.t.} \quad & x_1 - x_2 + x_3 = 1 \\
 & 2x_1 - x_2 + x_4 = 4 \\
 & x_i \geq 0 \quad \forall i = 1, \dots, 4.
 \end{aligned}$$

- The first iteration:

$$\begin{array}{cccc|c}
 -1 & 0 & 0 & 0 & 0 \\
 \hline
 \boxed{1} & -1 & 1 & 0 & x_3 = 1 \\
 2 & -1 & 0 & 1 & x_4 = 4
 \end{array}
 \rightarrow
 \begin{array}{cccc|c}
 0 & -1 & 1 & 0 & 1 \\
 \hline
 1 & -1 & 1 & 0 & x_1 = 1 \\
 0 & 1 & -2 & 1 & x_4 = 2
 \end{array}$$

Unbounded LPs

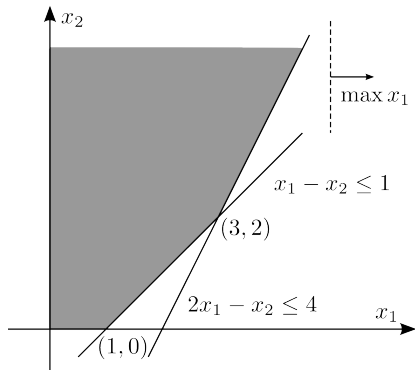
- ▶ The second iteration:

$$\begin{array}{cccc|c}
 0 & -1 & 1 & 0 & 1 \\
 \hline
 1 & -1 & 1 & 0 & x_1 = 1 \\
 0 & \boxed{1} & -2 & 1 & x_4 = 2
 \end{array}
 \rightarrow
 \begin{array}{cccc|c}
 0 & 0 & -1 & 1 & 3 \\
 \hline
 1 & 0 & -1 & 1 & x_1 = 3 \\
 0 & 1 & -2 & 1 & x_2 = 2
 \end{array}$$

- ▶ How may we do the third iteration? The **ratio test** fails!
 - ▶ Only rows with positive denominators participate in the ratio test.
 - ▶ Now all the denominators are nonpositive! Which variable to leave?
- ▶ No one should leave: Increasing x_3 makes x_1 and x_2 become larger.
 - ▶ Row 1: $x_1 - x_3 + x_4 = 3$.
 - ▶ Row 2: $x_2 - 2x_3 + x_4 = 2$.
- ▶ The direction is thus an **unbounded improving direction**.

Unbounded improving directions

- At $(3, 2)$, when we enter x_3 , we move along the rightmost edge. Geometrically, both nonbinding constraints $x_1 \geq 0$ and $x_2 \geq 0$ are “behind us”.



Detecting unbounded LPs

- ▶ For a **minimization** LP, whenever we see **any** column in **any** tableau

	\bar{c}_j
d_1	
\vdots	
d_m	

such that $\bar{c}_j > 0$ and $d_i \leq 0$ for all $i = 1, \dots, m$, we may stop and conclude that this LP is unbounded.

- ▶ $\bar{c}_j > 0$: This is an improving direction.
- ▶ $d_i \leq 0$ for all $i = 1, \dots, m$: This is an unbounded direction.
- ▶ What is the unbounded condition for a **maximization** problem?

Multiple optimal solutions

- ▶ Consider another example (in standard form directly):

$$\begin{array}{llllllll} \max & x_1 & + & x_2 & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & & = & 12 \\ & 2x_1 & + & x_2 & & & + & x_4 & = & 12 \\ & x_1 & + & x_2 & & & & & + & x_5 & = & 7 \\ & x_i & \geq & 0 & \forall i = & 1, \dots, 5. & & & & & & \end{array}$$

Multiple optimal solutions

- In two iterations, we find an optimal solution:

$$\begin{array}{c}
 \begin{array}{ccccc|c}
 -1 & -1 & 0 & 0 & 0 & 0 \\
 \hline
 1 & 2 & 1 & 0 & 0 & x_3 = 12 \\
 \boxed{2} & 1 & 0 & 1 & 0 & x_4 = 12 \\
 1 & 1 & 0 & 0 & 1 & x_5 = 7
 \end{array}
 & \rightarrow &
 \begin{array}{ccccc|c}
 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 6 \\
 \hline
 0 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 & x_3 = 6 \\
 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & x_1 = 6 \\
 0 & \boxed{\frac{1}{2}} & 0 & -\frac{1}{2} & 1 & x_5 = 1
 \end{array} \\
 \\
 & & & \rightarrow & &
 \begin{array}{ccccc|c}
 0 & 0 & 0 & 0 & 1 & 7 \\
 \hline
 0 & 0 & 1 & 1 & -2 & x_3 = 3 \\
 1 & 0 & 0 & 1 & -2 & x_1 = 5 \\
 0 & 1 & 0 & -1 & 2 & x_2 = 2
 \end{array}
 \end{array}$$

Multiple optimal solutions

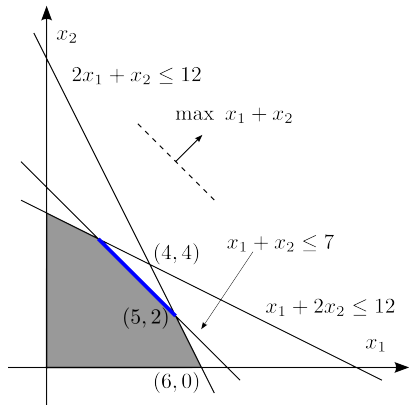
- ▶ In practice, we will simply stop and report the optimal solution.
- ▶ But here the optimal tableau shows the existence of **multiple** optimal solutions.

$$\begin{array}{cccc|c}
 0 & 0 & 0 & \mathbf{0} & 1 & 7 \\
 \hline
 0 & 0 & 1 & 1 & -2 & x_3 = 3 \\
 1 & 0 & 0 & 1 & -2 & x_1 = 5 \\
 0 & 1 & 0 & -1 & 2 & x_2 = 2
 \end{array}$$

- ▶ What does a zero reduced cost mean?
 - ▶ When we increase x_4 , z will not be affected.
- ▶ As the current solution is optimal, if there is a direction such that moving along it does not change the objective value, **all points** along that direction are optimal.

Multiple optimal solutions

- ▶ At **an** optimal solution $(5, 2)$, by entering x_4 , we move along $x_1 + x_2 = 7$. All points on that edge are optimal.
- ▶ For a nondegenerate LP, at an **optimal** tableau, if a nonbasic variable x_j has a **zero reduced cost**, the LP has multiple optimal solutions.
 - ▶ For a degenerate LP (to be discussed later in this lecture), the condition is not sufficient.
 - ▶ In practice, knowing this is not very valuable.



Road map

- ▶ Information on tableaus.
- ▶ **Finding an initial bfs.**
- ▶ Degeneracy and efficiency.
- ▶ The matrix way of doing simplex.

Feasibility of an LP

- ▶ When an LP

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \leq b \\ & x \geq 0 \end{aligned}$$

satisfies $b \geq 0$, finding a bfs for its standard form

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax + Iy = b \\ & x, y \geq 0, \end{aligned}$$

is trivial.

- ▶ We may form a feasible basis with all the slack variables y .
- ▶ What if there are some “=” or “ \geq ” constraints?

Feasibility of an LP

- For example, given an LP

$$\begin{array}{ll}
 \min & x_1 \\
 \text{s.t.} & x_1 + x_2 - x_3 + x_4 \geq 10 \\
 & 3x_1 + 2x_2 + 9x_3 - x_4 = 10 \\
 & x_1 - 8x_2 + 2x_3 - 6x_4 \leq 10 \\
 & x_i \geq 0 \quad \forall i = 1, \dots, 4
 \end{array}$$

whose standard form is

$$\begin{array}{ll}
 \min & x_1 \\
 \text{s.t.} & x_1 + x_2 - x_3 + x_4 - x_5 = 10 \\
 & 3x_1 + 2x_2 + 9x_3 - x_4 = 10 \\
 & x_1 - 8x_2 + 2x_3 - 6x_4 + x_6 = 10 \\
 & x_i \geq 0 \quad \forall i = 1, \dots, 6,
 \end{array}$$

it is nontrivial to find a feasible basis (if there is one).

The two-phase implementation

- ▶ To find an initial bfs (or show that there is none), we may apply the **two-phase implementation**.
- ▶ Given a standard form LP (P) , we construct a **phase-I LP** (Q) :¹

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array}$$

$$(Q) \quad \begin{array}{ll} \min & 1^T y \\ \text{s.t.} & Ax + Iy = b \\ & x, y \geq 0. \end{array}$$

- ▶ (Q) has a trivial bfs $(x, y) = (0, b)$, so we can apply the simplex method on (Q) . But so what?

Proposition 1

(P) is feasible if and only if (Q) has an optimal bfs $(x, y) = (\bar{x}, 0)$. In this case, \bar{x} is a bfs of (P) .

¹Even if in (P) we have a maximization objective function, (Q) is still the same.

The two-phase implementation

- ▶ After we solve (Q) , either we know (P) is infeasible or we have a feasible basis of (P) .
- ▶ In the latter case, we can recover the objective function of the original (P) to get a **phase-II LP**.
 - ▶ “The phase-II LP” is nothing but the original (P) .
 - ▶ Phase I for a **feasible** solution and phase II for an **optimal** solution.
- ▶ Regarding those added variables:
 - ▶ They are **artificial variables** and have no physical meaning. They are created only for checking feasibility.
 - ▶ If a constraint already has a variable that can be included in a trivial basis, we do not need to add an artificial variable in that constraint.
 - ▶ This happens to those “ \leq ” constraints (if the RHS is nonnegative).
- ▶ We then **adjust** the tableau according to the initial basis and **continue** applying the simplex method on the phase-II LP.

Example 1: Phase I

- ▶ Consider an LP

$$\begin{aligned}
 \max \quad & x_1 + x_2 \\
 \text{s.t.} \quad & 2x_1 + x_2 \geq 6 \\
 & x_1 + 2x_2 \leq 6 \\
 & x_i \geq 0 \quad \forall i = 1, 2.
 \end{aligned}$$

which has no trivial bfs (due to the “ \geq ” constraint).

- ▶ Its Phase-I standard form LP is

$$\begin{aligned}
 \min \quad & x_5 \\
 \text{s.t.} \quad & 2x_1 + x_2 - x_3 + x_5 = 6 \\
 & x_1 + 2x_2 + x_4 = 6 \\
 & x_i \geq 0 \quad \forall i = 1, \dots, 5.
 \end{aligned}$$

- ▶ We need only one artificial variable x_5 . x_3 and x_4 are slack variables.

Example 1: preparing the initial tableau

- ▶ Let's try to solve the Phase-I LP. First, let's prepare the initial tableau:

$$\begin{array}{ccccc|c}
 0 & 0 & 0 & 0 & -1 & 0 \\
 \hline
 2 & 1 & -1 & 0 & 1 & x_5 = 6 \\
 1 & 2 & 0 & 1 & 0 & x_4 = 6
 \end{array}$$

- ▶ Is this a valid tableau? No!
 - ▶ For all basic columns (in this case, columns 4 and 5), the 0th row should contain 0.
 - ▶ So we need to first **adjust the 0th row** with elementary row operations.

Example 1: preparing the initial tableau

- ▶ Let's adjust row 0 by adding row 1 to row 0.

$$\begin{array}{cccc|c}
 0 & 0 & 0 & 0 & -1 & 0 \\
 \hline
 2 & 1 & -1 & 0 & 1 & x_5 = 6 \\
 1 & 2 & 0 & 1 & 0 & x_4 = 6
 \end{array}
 \quad \begin{array}{c} \text{adjust} \\ \underbrace{\rightarrow} \end{array}
 \quad \begin{array}{cccc|c}
 2 & 1 & -1 & 0 & 0 & 6 \\
 \hline
 2 & 1 & -1 & 0 & 1 & x_5 = 6 \\
 1 & 2 & 0 & 1 & 0 & x_4 = 6
 \end{array}$$

- ▶ Now we have a valid initial tableau to start from!
- ▶ The current bfs is $x^0 = (0, 0, 0, 6, 6)$, which corresponds to an **infeasible** solution to the original LP.
 - ▶ We know this because there are positive artificial variables.

Example 1: solving the Phase-I LP

- ▶ Solving the Phase-I LP takes only one iteration:

$$\begin{array}{cccc|c}
 2 & 1 & -1 & 0 & 0 & 6 \\
 \hline
 \boxed{2} & 1 & -1 & 0 & 1 & x_5 = 6 \\
 1 & 2 & 0 & 1 & 0 & x_4 = 6
 \end{array}
 \quad \rightarrow \quad
 \begin{array}{cccc|c}
 0 & 0 & 0 & 0 & 0 \\
 \hline
 1 & \frac{1}{2} & -\frac{1}{2} & 0 & & x_1 = 3 \\
 0 & \frac{3}{2} & \frac{1}{2} & 1 & & x_4 = 3
 \end{array}$$

- ▶ Whenever an artificial variable leaves the basis, we will not need to enter it again. Therefore, we may remove that column to save calculations.
- ▶ As we can remove all artificial variables, the original LP is feasible.
- ▶ A feasible basis for the original LP is $\{x_1, x_4\}$.

Example 1: solving the Phase-II LP

- ▶ Now let's construct the Phase-II LP.
- ▶ Step 1: put the original objective function “max $x_1 + x_2$ ” back:

$$\begin{array}{cccc|c}
 -1 & -1 & 0 & 0 & 0 \\
 \hline
 1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\
 0 & \frac{3}{2} & \frac{1}{2} & 1 & x_4 = 3
 \end{array}$$

- ▶ Is this a valid tableau? No!
 - ▶ Column 1, which should be basic, contains a nonzero number in the 0th row. It must be adjusted to 0.
- ▶ Before we run iterations, let's adjust the 0th row again.

Example 1: solving the Phase-II LP

- Let's fix the 0th row and then run two iterations.

$$\begin{array}{cccc|c} -1 & -1 & 0 & 0 & 0 \\ \hline 1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\ 0 & \frac{3}{2} & \frac{1}{2} & 1 & x_4 = 3 \end{array}$$

adjust
→

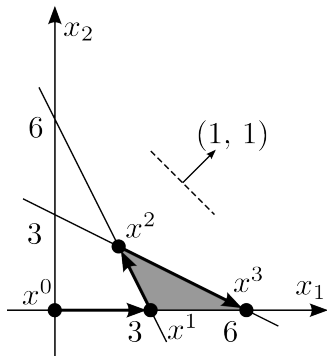
$$\begin{array}{cccc|c} 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & 3 \\ \hline 1 & \frac{1}{2} & -\frac{1}{2} & 0 & x_1 = 3 \\ 0 & \boxed{\frac{3}{2}} & \frac{1}{2} & 1 & x_4 = 3 \end{array}$$

$$\rightarrow \begin{array}{cccc|c} 0 & 0 & -\frac{1}{3} & \frac{1}{3} & 4 \\ \hline 1 & 0 & -\frac{2}{3} & -\frac{1}{3} & x_1 = 2 \\ 0 & 1 & \boxed{\frac{1}{3}} & \frac{2}{3} & x_2 = 2 \end{array}$$

$$\rightarrow \begin{array}{cccc|c} 0 & 1 & 0 & 1 & 6 \\ \hline 1 & 2 & 0 & 1 & x_1 = 6 \\ 0 & 3 & 1 & 2 & x_3 = 6 \end{array}$$

- The optimal bfs is $(6, 0, 6, 0)$.

Example 1: visualization



- ▶ x^0 is infeasible (the artificial variable x_5 is positive).
- ▶ x^1 is the initial bfs (as a result of Phase I).
- ▶ x^3 is the optimal bfs (as a result of Phase II).

Example 2: Phase-I LP

- ▶ Consider another LP

$$\begin{aligned}
 \max \quad & x_1 + x_2 \\
 \text{s.t.} \quad & 2x_1 + x_2 \geq 6 \\
 & x_1 + 2x_2 = 6 \\
 & x_i \geq 0 \quad \forall i = 1, 2
 \end{aligned}$$

and its Phase-I LP

$$\begin{aligned}
 \min \quad & x_4 + x_5 \\
 \text{s.t.} \quad & 2x_1 + x_2 - x_3 + x_4 = 6 \\
 & x_1 + 2x_2 + x_5 = 6 \\
 & x_i \geq 0 \quad \forall i = 1, \dots, 5.
 \end{aligned}$$

- ▶ Please note that there are two artificial variables x_4 and x_5 (why?).
 - ▶ How about x_3 ?

Example 2: solving the Phase-I LP

- We first fix the 0th row and then run two iterations to remove all the artificial variables:

$$\begin{array}{ccccc|c}
 0 & 0 & 0 & -1 & -1 & 0 \\
 \hline
 2 & 1 & -1 & 1 & 0 & x_4 = 6 \\
 1 & 2 & 0 & 0 & 1 & x_5 = 6
 \end{array}
 \xrightarrow{\text{adjust}}
 \begin{array}{ccccc|c}
 3 & 3 & -1 & 0 & 0 & 12 \\
 \hline
 \boxed{2} & 1 & -1 & 1 & 0 & x_4 = 6 \\
 1 & 2 & 0 & 0 & 1 & x_5 = 6
 \end{array}$$

$x^0 = (0, 0, 0, \underline{6}, \underline{6})$ is infeasible

$$\begin{array}{ccccc|c}
 0 & \frac{3}{2} & \frac{1}{2} & 0 & & 3 \\
 \hline
 1 & \frac{1}{2} & -\frac{1}{2} & 0 & & x_1 = 3 \\
 0 & \boxed{\frac{3}{2}} & \frac{1}{2} & 1 & & x_5 = 3
 \end{array}
 \rightarrow
 \begin{array}{ccc|c}
 0 & 0 & 0 & 0 \\
 \hline
 1 & 0 & -\frac{2}{3} & x_1 = 2 \\
 0 & 1 & \frac{1}{3} & x_2 = 2
 \end{array}$$

$x^1 = (3, 0, 0, \underline{0}, \underline{3})$ is infeasible

$x^2 = (2, 2, 0, \underline{0}, \underline{0})$ is feasible

Example 2: solving the Phase-II LP

- With the initial basis $\{x_1, x_2\}$, we then solve the Phase-II LP in one iteration (do not forget to fix the 0th row).²

$$\begin{array}{ccc|c}
 -1 & -1 & 0 & 0 \\
 \hline
 1 & 0 & -\frac{2}{3} & x_1 = 2 \\
 0 & 1 & \frac{1}{3} & x_2 = 2
 \end{array}
 \quad \begin{array}{c} \text{adjust} \\ \rightarrow \end{array}
 \quad \begin{array}{ccc|c}
 0 & 0 & -\frac{1}{3} & 4 \\
 \hline
 1 & 0 & -\frac{2}{3} & x_1 = 2 \\
 0 & 1 & \boxed{\frac{1}{3}} & x_2 = 2
 \end{array}$$

$x^2 = (2, 2, 0)$ is not optimal

$$\begin{array}{ccc|c}
 0 & 1 & 0 & 6 \\
 \hline
 1 & 2 & 0 & x_1 = 6 \\
 0 & 3 & 1 & x_3 = 6
 \end{array}
 \rightarrow$$

$x^3 = (6, 0, 6)$ is optimal

²Would you visualize the whole process by yourself?

Example 3: Phase-I LP

- Consider the LP

$$\begin{array}{ll} \max & x_1 \\ \text{s.t.} & 2x_1 + x_2 \leq 4 \\ & x_1 + x_2 = 6 \\ & x_i \geq 0 \quad \forall i = 1, 2 \end{array}$$

and its Phase-I LP

$$\begin{array}{ll} \min & x_4 \\ \text{s.t.} & 2x_1 + x_2 + x_3 = 4 \\ & x_1 + x_2 + x_4 = 6 \\ & x_i \geq 0 \quad \forall i = 1, \dots, 4. \end{array}$$

Example 3: solving the Phase-I LP

- After adjusting the 0th row, we run two iterations:

$$\begin{array}{c|c}
 0 & 0 & 0 & -1 & 0 \\
 \hline
 2 & 1 & 1 & 0 & x_3 = 4 \\
 1 & 1 & 0 & 1 & x_4 = 6
 \end{array}
 \quad \xrightarrow{\text{adjust}} \quad
 \begin{array}{c|c}
 1 & 1 & 0 & 0 & 6 \\
 \hline
 \boxed{2} & 1 & 1 & 0 & x_3 = 4 \\
 1 & 1 & 0 & 1 & x_4 = 6
 \end{array}$$

$x^0 = (0, 0, 4, \underline{6})$ is infeasible

$$\begin{array}{c|c}
 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 4 \\
 \hline
 1 & \boxed{\frac{1}{2}} & \frac{1}{2} & 0 & x_1 = 2 \\
 0 & \frac{1}{2} & -\frac{1}{2} & 1 & x_4 = 4
 \end{array}
 \quad \rightarrow \quad
 \begin{array}{c|c}
 -1 & 0 & -1 & 0 & 2 \\
 \hline
 2 & 1 & 1 & 0 & x_2 = 4 \\
 -1 & 0 & -1 & 1 & x_4 = 2
 \end{array}$$

$x^1 = (0, 2, 0, \underline{4})$ is infeasible

$x^2 = (0, 4, 0, \underline{2})$ is infeasible

Example 3: solving the Phase-I LP

- ▶ The final tableau

$$\begin{array}{cccc|c}
 -1 & 0 & -1 & 0 & 2 \\
 \hline
 2 & 1 & 1 & 0 & x_2 = 4 \\
 -1 & 0 & -1 & 1 & x_4 = 2
 \end{array}$$

is optimal (for the Phase-I LP).

- ▶ However, in the Phase-I optimal solution $(0, 4, 0, 2)$, the artificial variable x_4 is still in the basis (and positive).
- ▶ Therefore, we conclude that the original LP is infeasible.³

³Try to visualize this!

Road map

- ▶ Information on tableaus.
- ▶ Finding an initial bfs.
- ▶ **Degeneracy and efficiency.**
- ▶ The matrix way of doing simplex.

Degeneracy

- ▶ Recall that an LP is **degenerate** if multiple bases correspond to a single basic solution.
- ▶ As an example, consider the following LP

$$\begin{aligned}
 \max \quad & x_1 + 3x_2 \\
 \text{s.t.} \quad & x_1 + x_2 \leq 3 \\
 & 2x_1 + 3x_2 \leq 6 \\
 & x_i \geq 0 \quad \forall i = 1, 2
 \end{aligned}$$

and its standard form

$$\begin{aligned}
 \max \quad & x_1 + 3x_2 \\
 \text{s.t.} \quad & x_1 + x_2 + x_3 = 3 \\
 & 2x_1 + 3x_2 + x_4 = 6 \\
 & x_i \geq 0 \quad \forall i = 1, \dots, 4.
 \end{aligned}$$

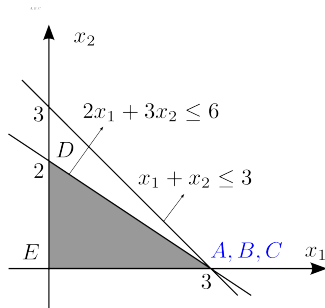
Degeneracy

- ▶ The six bases of

$$\begin{array}{rcll}
 \max & x_1 & + & 3x_2 \\
 \text{s.t.} & x_1 & + & x_2 + x_3 = 3 \\
 & 2x_1 & + & 3x_2 + x_4 = 6 \\
 & x_i & \geq & 0 \quad \forall i = 1, \dots, 4
 \end{array}$$

correspond to four distinct basic solutions.

Basis	Extreme point	Basic solution			
		x_1	x_2	x_3	x_4
$\{x_1, x_2\}$	A	3	0	0	0
$\{x_1, x_3\}$	B	3	0	0	0
$\{x_1, x_4\}$	C	3	0	0	0
$\{x_2, x_3\}$	D	0	2	1	0
$\{x_2, x_4\}$	–	0	3	0	–3
$\{x_3, x_4\}$	E	0	0	3	6



Impact of degeneracy

- ▶ In a degenerate LP, multiple feasible bases correspond to the same bfs.
- ▶ For the simplex method, it is possible to move to **another** basis but still at the **same** bfs.
- ▶ Running an iteration may have **no improvement!**
- ▶ Let's run the simplex method on this example.

Solving degenerate LPs

- ▶ After three iterations, we find an optimal solution:

$$\begin{array}{c|c}
 -1 & -3 & 0 & 0 & 0 \\
 \hline
 \boxed{1} & 1 & 1 & 0 & x_3 = 3 \\
 2 & 3 & 0 & 1 & x_4 = 6
 \end{array}
 \rightarrow
 \begin{array}{c|c}
 0 & -2 & 1 & 0 & 3 \\
 \hline
 1 & 1 & 1 & 0 & x_1 = 3 \\
 0 & \boxed{1} & -2 & 1 & x_4 = 0
 \end{array}$$

$$\rightarrow
 \begin{array}{c|c}
 0 & 0 & -3 & 2 & 3 \\
 \hline
 1 & 0 & \boxed{3} & -1 & x_1 = 3 \\
 0 & 1 & -2 & 1 & x_2 = 0
 \end{array}
 \rightarrow
 \begin{array}{c|c}
 1 & 0 & 0 & 1 & 6 \\
 \hline
 \frac{1}{3} & 0 & 1 & -\frac{1}{3} & x_3 = 1 \\
 \frac{2}{3} & 1 & 0 & \frac{1}{3} & x_2 = 2
 \end{array}$$

- ▶ In the second iteration, there is no improvement!
- ▶ The basis changes but the bfs does not change.

Efficiency of the simplex method

- ▶ In general, when we use the simplex method to solve a degenerate LP, there may be some iterations that have no improvements.
 - ▶ That may happen when multiple rows win the ratio test **at the same time**; those basic variables become 0 simultaneously.
- ▶ For some (very strange) instances, the simplex method needs to travel through **all** the bfs before it can make a conclusion.
- ▶ Therefore, the simplex method is an **exponential-time** algorithm.⁴
 - ▶ It may take an unacceptable long time to solve an LP.
- ▶ There are polynomial-time algorithms for Linear Programming.
 - ▶ For many practical problems, the simplex method is still faster.
- ▶ The simplex method is the most popular method for LP in industry.

⁴The number of iteration is $O(\binom{n}{m})$.

Efficiency of the simplex method

- ▶ When using the simplex method to solve an (original) LP, the number of **functional constraints** (m) greatly affects the computation time.
 - ▶ The computation time is roughly $O(m^3)$: proportional to the **cube** of the number of functional constraints.
 - ▶ Intuition: Number of iterations is $O(m)$ and number of operations in an iteration is $O(m^2)$.
- ▶ The number of variables, on the contrary, is not so important.
 - ▶ We calculate $x_B = A_B^{-1}b$ in each iteration, and $A_B \in \mathbb{R}^{m \times m}$.
- ▶ The **sparsity** of the coefficient matrix A is also important.
 - ▶ A is sparse means it has many zeros.
 - ▶ Practical problems typically have sparse coefficient matrices.
- ▶ For more information, see Chapters 5 and 7 (which will not be covered in this course).

Cycling

- ▶ One thing is even worse than running for a long time.
- ▶ At a degenerate bfs, the simplex method may enter an infinite loop! This is called **cycling**.
 - ▶ Basis 1 \rightarrow basis 2 \rightarrow basis 3 $\rightarrow \dots \rightarrow$ basis 1.
- ▶ This may happen when we use a “not so good” way of selecting entering and leaving variables.
 - ▶ If we select the nonbasic variable with the “most significant reduced cost”, cycling may occur.
- ▶ There are at least two ways to avoid cycling:
 - ▶ Randomize the selection of variables.
 - ▶ Apply an **anti-cycling** variable selection rule.

The smallest index rule

- ▶ One anti-cycling rule is the **smallest index rule**.⁵

Proposition 2 (The smallest index rule)

Using the following rule guarantees to solve a minimization LP in finite steps:

- ▶ *Among nonbasic variables with positive reduced costs, pick the one with the smallest index to enter the basis.*
 - ▶ *Among basic variables that have the smallest valid ratios, pick the one with smallest index to exist.*
- ▶ The smallest index rule may not generate the **least iterations** toward an optimal solution.
 - ▶ No variable selection rule can guarantee to be the most efficient!
 - ▶ The smallest index rule can guarantee **no cycling**!

⁵Developed by Bland in 1977.

Road map

- ▶ Information on tableaus.
- ▶ Finding an initial bfs.
- ▶ Degeneracy and efficiency.
- ▶ **The matrix way of doing simplex.**

Implementation of the simplex method

- ▶ When one implements the simplex method with computer programs, using tableaus is not the most efficient way.
- ▶ Using **matrices** is the most efficient.
- ▶ Recall that the standard form LP can be expressed as

$$\begin{aligned} \min \quad & c_B^T A_B^{-1} b - (c_B^T A_B^{-1} A_N - c_N^T) x_N \\ \text{s.t.} \quad & x_B = A_B^{-1} b - A_B^{-1} A_N x_N \\ & x_B, x_N \geq 0 \end{aligned}$$

or

$$\begin{aligned} z \quad & + (c_B^T A_B^{-1} A_N - c_N^T) x_N = c_B^T A_B^{-1} b \\ I x_B \quad & + A_B^{-1} A_N x_N = A_B^{-1} b. \end{aligned}$$

- ▶ We may do **matrix operations** to do iterations.

At any feasible basis

$$z + (c_B^T A_B^{-1} A_N - c_N) x_N = c_B^T A_B^{-1} b$$

$$I x_B + A_B^{-1} A_N x_N = A_B^{-1} b.$$

- ▶ At any feasible basis B :
 - ▶ The current bfs is $x = (x_B, x_N) = (A_B^{-1} b, 0)$ and the current $z = c_B^T A_B^{-1} b$.
- ▶ For the entering variable:
 - ▶ The **reduced costs** are $\bar{c}_N = c_B^T A_B^{-1} A_N - c_N^T$.
 - ▶ The reduced cost of variable x_j is $\bar{c}_j = c_B^T A_B^{-1} A_j - c_j$ for all $j \in N$.
 - ▶ If there exists $j \in N$ such that $\bar{c}_j > 0$, x_j may enter.
- ▶ For the leaving variable:
 - ▶ If x_j enters, the **ratio test** is to compare the ratios $\frac{(A_B^{-1} b)_i}{(A_B^{-1} A_j)_i}$.
 - ▶ The basic variable corresponding to row i may leave if $(A_B^{-1} A_j)_i > 0$ and

$$\frac{(A_B^{-1} b)_i}{(A_B^{-1} A_j)_i} \leq \frac{(A_B^{-1} b)_k}{(A_B^{-1} A_j)_k} \quad \forall k = 1, \dots, m \text{ such that } (A_B^{-1} A_j)_k > 0.$$

When we stop

- ▶ At any optimal basis B , we know that
 - ▶ The reduced costs $\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T \leq 0$.
 - ▶ The optimal bfs is $x^* = (x_B^*, x_N^*) = (A_B^{-1} b, 0)$.
 - ▶ The current objective value is $z^* = c_B^T A_B^{-1} b$.
- ▶ To detect multiple optimal solutions:
 - ▶ $\bar{c}_N^T = c_B^T A_B^{-1} A_N - c_N^T \leq 0$.
 - ▶ There exists $j \in N$ such that $\bar{c}_j = 0$.
- ▶ To detect unboundedness:
 - ▶ There exists $j \in N$ such that $\bar{c}_j > 0$.
 - ▶ Moreover, $(A_B^{-1} A_j)_i \leq 0$ for all $i \in B$.

Example

- Consider the example again:

$$\begin{array}{rcll}
 \min & -x_1 & & \\
 \text{s.t.} & 2x_1 & - x_2 & + x_3 = 4 \\
 & 2x_1 & + x_2 & + x_4 = 8 \\
 & & x_2 & + x_5 = 3 \\
 & x_i & \geq 0 & \forall i = 1, \dots, 5.
 \end{array}$$

- In the matrix representation, we have

$$c^T = [-1 \quad 0 \quad 0 \quad 0 \quad 0],$$

$$A = \begin{bmatrix} 2 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and } b = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix}.$$

A feasible basis

- Given $x_B = (x_1, x_4, x_5)$ and $x_N = (x_2, x_3)$, we have

$$c_B^T = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, \quad c_N^T = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$A_B = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix}.$$

- Given the basis, we have

$$x_B = A_B^{-1}b = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_4 \\ x_5 \end{bmatrix}, \quad \text{and}$$

$$z = c_B^T A_B^{-1}b = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = -2.$$

- The current bfs is $x = (x_1, x_2, x_3, x_4, x_5) = (2, 0, 0, 4, 3)$.

A feasible basis

- ▶ For $x_N = (x_2, x_3)$, the reduced costs are

$$\begin{aligned}\bar{c}_N &= c_B^T A_B^{-1} A_N - c_N^T \\ &= \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.\end{aligned}$$

- ▶ x_2 enters. For $x_B = (x_1, x_4, x_5)$, we have

- ▶ $A_B^{-1} A_2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ 1 \end{bmatrix}$ and $A_B^{-1} b = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$.
- ▶ $\frac{4}{2} < \frac{3}{1}$, so x_4 leaves.

An optimal basis

- ▶ Given $x_B = (x_1, x_2, x_5)$ and $x_N = (x_3, x_4)$ we have

$$c_B^T = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix}, \quad c_N^T = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

$$A_B = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad A_N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix}.$$

- ▶ Given the basis, we have

$$x_B = A_B^{-1}b = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix}, \quad \text{and}$$

$$z = c_B^T A_B^{-1}b = \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = -3.$$

- ▶ The current bfs is $x = (x_1, x_2, x_3, x_4, x_5) = (3, 2, 0, 0, 1)$.

An optimal basis

- ▶ For $x_N = (x_3, x_4)$, the reduced costs are

$$\begin{aligned}\bar{c}_N^T &= c_B^T A_B^{-1} A_N - c_N^T \\ &= \begin{bmatrix} -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}.\end{aligned}$$

- ▶ No variable should enter: This bfs is optimal.

The matrix way

- ▶ In short, the simplex method may be run with matrix calculations.
- ▶ In this way, the bottleneck is the calculation of A_B^{-1} .
- ▶ Nevertheless, because the current basis B and the previous one have only **one variable** different, the current A_B and the previous one have only **one column** different.
 - ▶ Calculating A_B^{-1} can be faster with the previous one.⁶
- ▶ In fact, how do you know that A_B is still **invertible** after changing one column?

⁶Section 5.4 contains relevant discussion about calculating A_B^{-1} .