

# Operations Research

## Linear Programming Duality

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# Introduction

- ▶ For business, we study how to formulate LPs.
- ▶ For engineering, we study how to solve LPs.
- ▶ For science, we study mathematical **properties** of LPs.
  - ▶ We will study **Linear Programming duality**.
  - ▶ It still has important applications.

# Road map

- ▶ **Primal-dual pairs.**
- ▶ Duality theorems.
- ▶ Shadow prices.

## Upper bounds of a maximization LP

- ▶ Consider the following LP

$$\begin{aligned} z^* = \max \quad & 4x_1 + 5x_2 + 8x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\ & 2x_1 + x_2 + 2x_3 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

- ▶ Suppose the LP is very hard to solve.
- ▶ Your friend proposes a solution  $\hat{x} = (\frac{1}{2}, 1, 1)$  with  $\hat{z} = 15$ .
  - ▶ If we know  $z^*$ , we may compare  $\hat{z}$  with  $z^*$ .
  - ▶ How to evaluate the performance of  $\hat{x}$  **without** solving the LP?
- ▶ If we can find an **upper bound** of  $z^*$ , that works!
  - ▶  $z^*$  cannot be greater than the upper bound.
  - ▶ So if  $\hat{z}$  is close to the upper bound,  $\hat{x}$  is quite good.<sup>1</sup>

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<sup>1</sup>You know 97 is quite high without knowing the highest in this class.

## Upper bounds of a maximization LP

- ▶ How to find an upper bound of  $z^*$  for

$$\begin{aligned} z^* = \max \quad & 4x_1 + 5x_2 + 8x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\ & 2x_1 + x_2 + 2x_3 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0? \end{aligned}$$

- ▶ How about this: Multiply the first constraint by 2, multiply the second constraint by 1, and then add them together:

$$\begin{aligned} 2(x_1 + 2x_2 + 3x_3) + (2x_1 + x_2 + 2x_3) &\leq 2 \times 6 + 4 \\ \Leftrightarrow 4x_1 + 5x_2 + 8x_3 &\leq 16. \end{aligned}$$

- ▶ Compare this with the objective function, we know  $z^* \leq 16$ .
  - ▶ Maybe  $z^*$  is exactly 16 (and the upper bound is **tight**). However, we do not know it here.
  - ▶  $\hat{z} = 15$  is close to  $z^* = 16$ , so  $\hat{x}$  is quite good.

## Upper bounds of a maximization LP

- ▶ How to find an upper bound of  $z^*$  for this one?

$$\begin{aligned} z^* = \max \quad & 3x_1 + 4x_2 + 8x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\ & 2x_1 + x_2 + 2x_3 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$

- ▶ 16 is also an upper bound:

$$\begin{aligned} & 3x_1 + 4x_2 + 8x_3 \\ & \leq 4x_1 + 5x_2 + 8x_3 \quad (\text{because } x_1 \geq 0, x_2 \geq 0) \\ & = 2(x_1 + 2x_2 + 3x_3) + (2x_1 + x_2 + 2x_3) \\ & \leq 2 \times 6 + 4 = 16. \end{aligned}$$

- ▶ It is quite likely that 16 is not a tight upper bound and there is a better one. How to improve our upper bound?

## Better upper bounds?

$$\begin{aligned}
 z^* = \max \quad & 3x_1 + 4x_2 + 8x_3 \\
 \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\
 & 2x_1 + x_2 + 2x_3 \leq 4 \\
 & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.
 \end{aligned}$$

- ▶ Changing **coefficients** multiplied on the two constraints modifies the proposed upper bound.
  - ▶ Different coefficients result in different **linear combinations**.
- ▶ Let's call the two coefficients  $y_1$  and  $y_2$ , respectively:

$$\begin{array}{rcl}
 x_1 + & 2x_2 + & 3x_3 \leq 6 \quad (\times y_1) \\
 2x_1 + & x_2 + & 2x_3 \leq 4 \quad (\times y_2) \\
 \hline
 (y_1 + 2y_2)x_1 + & (2y_1 + y_2)x_2 + & (3y_1 + 2y_2)x_3 \leq 6y_1 + 4y_2
 \end{array}$$

- ▶ We need  $y_1 \geq 0$  and  $y_2 \geq 0$  to preserve the “ $\leq$ ”.
- ▶ When do we have  $z^* \leq 6y_1 + 4y_2$ ?





## Nonpositive or free variables

- ▶ Suppose variables are not all nonnegative:

$$\begin{aligned}
 z^* = \max \quad & 3x_1 + 4x_2 + 8x_3 \\
 \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \leq 6 \\
 & 2x_1 + x_2 + 2x_3 \leq 4 \\
 & x_1 \geq 0, \quad x_2 \leq 0, \quad x_3 \text{ urs.}
 \end{aligned}$$

- ▶ If we want

$$\leq (y_1 + 2y_2)x_1 + (2y_1 + y_2)x_2 + (3y_1 + 2y_2)x_3,$$

now we need

$$\begin{aligned}
 y_1 + 2y_2 &\geq 3 && \text{because } x_1 \geq 0, \\
 2y_1 + y_2 &\leq 4 && \text{because } x_2 \leq 0, \text{ and} \\
 3y_1 + 2y_2 &= 8 && \text{because } x_3 \text{ is free.}
 \end{aligned}$$

## Nonpositive or free variables

- ▶ So the primal and dual LPs are

$$\begin{array}{ll}
 \max & 3x_1 + 4x_2 + 8x_3 \\
 \text{s.t.} & x_1 + 2x_2 + 3x_3 \leq 6 \\
 & 2x_1 + x_2 + 2x_3 \leq 4 \\
 & x_1 \geq 0, x_2 \leq 0, x_3 \text{ urs.}
 \end{array}$$

and

$$\begin{array}{ll}
 \min & 6y_1 + 4y_2 \\
 \text{s.t.} & y_1 + 2y_2 \geq 3 \\
 & 2y_1 + y_2 \leq 4 \\
 & 3y_1 + 2y_2 = 8 \\
 & y_1 \geq 0, y_2 \geq 0.
 \end{array}$$

- ▶ Some observations:
  - ▶ Primal max  $\Rightarrow$  Dual min.
  - ▶ Primal objective  $\Rightarrow$  Dual RHS.
  - ▶ Primal RHS  $\Rightarrow$  Dual objective.
- ▶ Moreover:
  - ▶ Primal " $\geq 0$ " variable  $\Rightarrow$  Dual " $\geq$ " constraint.
  - ▶ Primal " $\leq 0$ " variable  $\Rightarrow$  Dual " $\leq$ " constraint.
  - ▶ Primal free variable  $\Rightarrow$  Dual " $=$ " constraint.
- ▶ What if we have " $\geq$ " or " $=$ " primal constraints?

## No-less-than and equality constraints

- ▶ Suppose constraints are not all “ $\leq$ ”:

$$\begin{aligned} z^* = \max \quad & 3x_1 + 4x_2 + 8x_3 \\ \text{s.t.} \quad & x_1 + 2x_2 + 3x_3 \geq 6 \\ & 2x_1 + x_2 + 2x_3 = 4 \\ & x_1 \geq 0, x_2 \leq 0, x_3 \text{ urs.} \end{aligned}$$

- ▶ To obtain

$$y_1(x_1 + 2x_2 + 3x_3) + y_2(2x_1 + x_2 + 2x_3) \leq 6y_1 + 4y_2,$$

we now need  $y_1 \leq 0$ .  $y_2$  can be of any sign (i.e., free).

## No-less-than and equality constraints

- So the primal and dual LPs are

$$\begin{array}{ll}
 \max & 3x_1 + 4x_2 + 8x_3 \\
 \text{s.t.} & x_1 + 2x_2 + 3x_3 \geq 6 \\
 & 2x_1 + x_2 + 2x_3 = 4 \\
 & x_1 \geq 0, x_2 \leq 0, x_3 \text{ urs.}
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ll}
 \min & 6y_1 + 4y_2 \\
 \text{s.t.} & y_1 + 2y_2 \geq 3 \\
 & 2y_1 + y_2 \leq 4 \\
 & 3y_1 + 2y_2 = 8 \\
 & y_1 \leq 0, y_2 \text{ urs.}
 \end{array}$$

- Some more observations:
- Primal “ $\leq$ ” constraint  $\Rightarrow$  Dual “ $\geq 0$ ” variable.
  - Primal “ $\geq$ ” constraint  $\Rightarrow$  Dual “ $\leq 0$ ” variable.
  - Primal “ $=$ ” constraint  $\Rightarrow$  Dual free variable.

## The general rule

- In general, if the primal LP is

$$\begin{array}{llllll} \max & c_1x_1 & + & c_2x_2 & + & c_3x_3 \\ \text{s.t.} & A_{11}x_1 & + & A_{12}x_2 & + & A_{13}x_3 & \geq & b_1 \\ & A_{21}x_1 & + & A_{22}x_2 & + & A_{23}x_3 & \leq & b_2 \\ & A_{31}x_1 & + & A_{32}x_2 & + & A_{33}x_3 & = & b_3 \\ & x_1 \geq 0, & x_2 \leq 0, & x_3 \text{ urs.}, & & & & \end{array}$$

its dual LP is

$$\begin{array}{llllll} \min & b_1y_1 & + & b_2y_2 & + & b_3y_3 \\ \text{s.t.} & A_{11}y_1 & + & A_{21}y_2 & + & A_{31}y_3 & \geq & c_1 \\ & A_{12}y_1 & + & A_{22}y_2 & + & A_{32}y_3 & \leq & c_2 \\ & A_{13}y_1 & + & A_{23}y_2 & + & A_{33}y_3 & = & c_3 \\ & y_1 \leq 0, & y_2 \geq 0, & y_3 \text{ urs.} & & & & \end{array}$$

- Note that the constraint coefficient matrix is “**transposed**”.

## Matrix representation

- In general, if the primal LP

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + c_3x_3 \\ \text{s.t.} \quad & A_{11}x_1 + A_{12}x_2 + A_{13}x_3 = b_1 \\ & A_{21}x_1 + A_{22}x_2 + A_{23}x_3 = b_2 \\ & A_{31}x_1 + A_{32}x_2 + A_{33}x_3 = b_3 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \end{aligned}$$

is in the **standard form**, its dual LP is

$$\begin{aligned} \min \quad & b_1y_1 + b_2y_2 + b_3y_3 \\ \text{s.t.} \quad & A_{11}y_1 + A_{21}y_2 + A_{31}y_3 \geq c_1 \\ & A_{12}y_1 + A_{22}y_2 + A_{32}y_3 \geq c_2 \\ & A_{13}y_1 + A_{23}y_2 + A_{33}y_3 \geq c_3. \end{aligned}$$

- In matrix representation:

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

and

$$\begin{aligned} \min \quad & y^T b \\ \text{s.t.} \quad & y^T A \geq c^T. \end{aligned}$$

## The dual LP for a minimization primal LP

- ▶ For a minimization LP, its dual LP is to **maximize** the **lower bound**.
- ▶ Rules for the directions of variables and constraints are **reversed**:

$$\begin{array}{ll}
 \min & 3x_1 + 4x_2 + 8x_3 \\
 \text{s.t.} & x_1 + 2x_2 + 3x_3 \geq 6 \\
 & 2x_1 + x_2 + 2x_3 \leq 4 \\
 & x_1 \geq 0, x_2 \leq 0, x_3 \text{ urs.}
 \end{array}
 \Rightarrow
 \begin{array}{ll}
 \max & 6y_1 + 4y_2 \\
 \text{s.t.} & y_1 + 2y_2 \leq 3 \\
 & 2y_1 + y_2 \geq 4 \\
 & 3y_1 + 2y_2 = 8 \\
 & y_1 \geq 0, y_2 \leq 0.
 \end{array}$$

- ▶ Note that

$$\begin{aligned}
 & 3x_1 + 4x_2 + 8x_3 \\
 \geq & (y_1 + 2y_2)x_1 + (2y_1 + y_2)x_2 + (3y_1 + 2y_2)x_3 \\
 \geq & (x_1 + 2x_2 + 3x_3)y_1 + (2x_1 + x_2 + 2x_3)y_2 \\
 \geq & 6y_1 + 4y_2.
 \end{aligned}$$

## The general rule, uniqueness, and symmetry

- ▶ The general rule for finding the dual LP:

Obj. function	max	min	Obj. function
Constraint	$\leq$	$\geq 0$	Variable
	$\geq$	$\leq 0$	
	$=$	urs.	
Variable	$\geq 0$	$\geq$	Constraint
	$\leq 0$	$\leq$	
	urs.	$=$	

- ▶ If the primal LP is a maximization problem, do it from left to right.
- ▶ If the primal LP is a minimization problem, do it from right to left.

### Proposition 1 (Uniqueness and symmetry of duality)

*For any primal LP, there is a unique dual, whose dual is the primal.*



## Examples of primal-dual pairs

► Example 1:

$$\begin{array}{ll}
 \min & 2x_1 + 3x_2 \\
 \text{s.t.} & 4x_1 + x_2 \leq 9 \\
 & x_1 \geq 6 \\
 & 2x_1 - x_2 \geq 8 \\
 & x_1 \leq 0, x_2 \text{ urs.}
 \end{array}
 \Leftrightarrow
 \begin{array}{ll}
 \max & 9y_1 + 6y_2 + 8y_3 \\
 \text{s.t.} & 4y_1 + y_2 + 2y_3 \geq 2 \\
 & y_1 - y_3 = 3 \\
 & y_1 \leq 0, y_2 \geq 0, y_3 \geq 0.
 \end{array}$$

► Example 2:

$$\begin{array}{ll}
 \max & 3x_1 - x_2 \\
 \text{s.t.} & x_1 + 2x_2 = 6 \\
 & 3x_1 + 3x_2 \leq -4 \\
 & x_1 \text{ urs.}, x_2 \geq 0.
 \end{array}
 \Leftrightarrow
 \begin{array}{ll}
 \min & 6y_1 - 4y_2 \\
 \text{s.t.} & y_1 + 3y_2 = 3 \\
 & 2y_1 + 3y_2 \geq -1 \\
 & y_1 \text{ urs.}, y_2 \geq 0.
 \end{array}$$

# Road map

- ▶ Primal-dual pairs.
- ▶ **Duality theorems.**
- ▶ Shadow prices.

## Duality theorems

- ▶ Duality provides many interesting properties.
- ▶ We will illustrate these properties for standard form primal LPs:

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array} \quad (1)$$

- ▶ It can be shown that all the properties that we will introduce apply to other primal-dual pairs.

## Weak duality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- ▶ The dual LP provides an **upper bound** of the primal LP.

### Proposition 2 (Weak duality)

*For the LPs defined in (1), if  $x$  and  $y$  are primal and dual feasible, then  $c^T x \leq y^T b$ .*

*Proof.* As long as  $x$  and  $y$  are primal and dual feasible, we have

$$\begin{aligned} c^T x &\leq y^T Ax && (x \geq 0 \text{ and } y^T A \geq c^T) \\ &\leq y^T b && (Ax = b). \end{aligned}$$

Therefore, weak duality holds. □

## Sufficiency of optimality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- ▶ We now have a **sufficient condition** for optimal solutions.

### Proposition 3 (Sufficient condition for optimality)

*If  $\bar{x}$  and  $\bar{y}$  are primal and dual feasible and  $c^T \bar{x} = \bar{y}^T b$ , then  $\bar{x}$  and  $\bar{y}$  are primal and dual optimal.*

*Proof.* For all dual feasible  $y$ , we have  $c^T \bar{x} \leq y^T b$  by weak duality. But we are given that  $c^T \bar{x} = \bar{y}^T b$ , so we have  $\bar{y}^T b \leq y^T b$  for all dual feasible  $y$ . This just tells us that  $\bar{y}$  is dual optimal. For  $\bar{x}$  it is the same.  $\square$

- ▶ Given a primal feasible solution  $\bar{x}$ , if we can find a dual feasible solution so that their objective values are **identical**,  $\bar{x}$  is optimal.

## The dual optimal solution

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- If we have solved the primal LP, the **dual optimal solution** is there.

### Proposition 4 (Dual optimal solution)

*For the LPs defined in (1), if  $\bar{x}$  is primal optimal with basis  $B$ , then  $\bar{y}^T = c_B^T A_B^{-1}$  is dual optimal.*

*Proof.* Because  $B$  is optimal, the reduced costs  $c_B^T A_B^{-1} A_N - c_N^T \geq 0$ . As  $c_B^T = c_B^T A_B^{-1} A_B$ , we have

$$\bar{y}^T A = c_B^T A_B^{-1} A = c_B^T A_B^{-1} [ A_B \quad A_N ] \geq [ c_B^T \quad c_N^T ] = c^T$$

and thus  $\bar{y}$  is dual feasible. As  $\bar{y}^T b = c_B^T A_B^{-1} b = c_B^T x_B = c^T x$ ,  $\bar{x}$  and  $\bar{y}$  have the same objective value and are thus both optimal.  $\square$

## Strong duality

$$\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0. \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & y^T b \\ \text{s.t.} & y^T A \geq c^T. \end{array}$$

- The fact that  $c_B^T A_B^{-1}$  is dual optimal implies **strong duality**:

### Proposition 5 (Strong duality)

*For the LPs defined in (1),  $\bar{x}$  and  $\bar{y}$  are primal and dual optimal if and only if  $\bar{x}$  and  $\bar{y}$  are primal and dual feasible and  $c^T \bar{x} = \bar{y}^T b$ .*

*Proof.* To prove this if-and-only-if statement:

- ( $\Leftarrow$ ): By Proposition 3.
- ( $\Rightarrow$ ): As  $c_B^T A_B^{-1}$  is an dual optimal solution, the dual optimal objective value is  $c_B^T A_B^{-1} b$ , which equals the primal optimal objective value  $c^T \bar{x}$ .  
As  $\bar{y}$  is dual optimal,  $\bar{y}^T b = c_B^T A_B^{-1} b = c^T \bar{x}$ .<sup>2</sup> □

<sup>2</sup>As the dual LP may or may not have a unique optimal solution,  $\bar{y}$  and  $c_B^T A_B^{-1}$  may or may not be identical. In either case, the statement holds.

## Implications of strong duality

- ▶ Strong duality certainly implies weak duality.
  - ▶ Weak duality says that the dual LP provides a bound.
  - ▶ Strong duality says that the bound is **tight**, i.e., cannot be improved.
- ▶ The primal and dual LPs are **equivalent**.
- ▶ Given the result of one LP, we may predict the result of its dual:

Primal	Dual		
	Infeasible	Unbounded	Finitely optimal
Infeasible	✓	✓	×
Unbounded	✓	×	×
Finitely optimal	×	×	✓

- ▶ ✓ means possible, × means impossible.
- ▶ Primal unbounded  $\Rightarrow$  no upper bound  $\Rightarrow$  dual infeasible.
- ▶ Primal finitely optimal  $\Rightarrow$  finite objective value  $\Rightarrow$  dual finitely optimal.
- ▶ If primal is infeasible, the dual may still be infeasible (by examples).



## Complementary slackness

- ▶ Consider  $w$ , the **slack** variables of the dual LP:

$$\begin{aligned} \min \quad & y^T b \\ \text{s.t.} \quad & y^T A - w^T = c^T \\ & w \geq 0. \end{aligned} \tag{2}$$

### Proposition 6 (Complementary slackness)

*For the primal defined in (1) and dual defined in (2),  $\bar{x}$  and  $(\bar{y}, \bar{w})$  are primal and dual optimal if and only if  $\bar{w}^T \bar{x} = 0$ .*

*Proof.* We have  $c^T \bar{x} = (\bar{y}^T A - \bar{w}^T) \bar{x} = \bar{y}^T A \bar{x} - \bar{w}^T \bar{x} = \bar{y}^T b - \bar{w}^T \bar{x}$ . Therefore,  $\bar{w}^T \bar{x} = 0$  if and only if  $c^T \bar{x} = \bar{y}^T b$ , i.e.,  $\bar{x}$  and  $(\bar{y}, \bar{w})$  are primal and dual optimal according to strong duality.  $\square$

- ▶ Note that  $\bar{w}^T \bar{x} = 0$  if and only if  $\bar{w}_i \bar{x}_i = 0$  for all  $i$  as  $\bar{x} \geq 0$  and  $\bar{w} \geq 0$ .
- ▶ If a dual (respectively, primal) constraint is **nonbinding**, the corresponding primal (respectively, dual) variable is **zero**.

## Why duality?

- ▶ Why duality? Given an LP:
  - ▶ We may solve it directly.
  - ▶ Or we may solve the dual LP and then get the primal optimal solution (by Proposition 4).
- ▶ Why bothering?
- ▶ Recall that the computation time of the simplex method is roughly proportional to  $m^3$ .
  - ▶  $m$  is the number of functional constraints of the original LP.
  - ▶ And  $n$ , the number of variables of the original LP, does not matter a lot.
- ▶ If  $m \gg n$ , solving the dual LP can take a significantly **shorter time** than solving the primal!
- ▶ There are many other benefits for having duality. We will see some more in this course.
- ▶ Read Sections 6.1, 6.3, and 6.4 carefully.

# Road map

- ▶ Primal-dual pairs.
- ▶ Duality theorems.
- ▶ **Shadow prices.**

## A product mix problem

- ▶ Suppose we produce tables and chairs with wood and labors. In total we have six units of wood and six labor hours.
  - ▶ Each table is sold at \$3 and requires 2 units of wood and 1 labor hour.
  - ▶ Each chair is sold at \$1 and requires 1 unit of wood and 2 labor hours.

How may we formulate an LP to maximize our sales revenue?

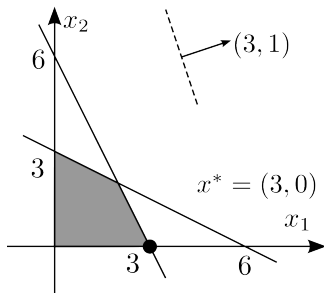
- ▶ The formulation is

$x_1$  = number of tables produced

$x_2$  = number of chairs produced.

$$\begin{array}{ll} \max & 3x_1 + x_2 \\ \text{s.t.} & 2x_1 + x_2 \leq 6 \\ & x_1 + 2x_2 \leq 6 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

- ▶ The optimal solution is  $x^* = (3, 0)$ .



## “What-if” questions

- ▶ In practice, people often ask “**what-if**” questions:
  - ▶ What if the unit price of chairs becomes \$2?
  - ▶ What if each table requires 3 unit of wood?
  - ▶ What if we have 10 units of wood?
- ▶ Why what-if questions?
  - ▶ Parameters may fluctuate.
  - ▶ Estimation of parameters may be inaccurate.
  - ▶ Looking for ways to improve the business.
- ▶ For realistic problems, what-if questions can be hard.
  - ▶ Even though it may be just a tiny modification of one parameter, the optimal solution may change a lot.
- ▶ The tool for answering what-if questions is **sensitivity analysis**.

# Humboldt Redwood



- ▶ Pacific Lumber Company (now Humboldt Redwood) has over 200,000 acres of forests and five mills in Humboldt County.
- ▶ **Sustainability** is important in making operational decisions.
- ▶ They contracted with an OR team to develop a 120-year forest ecosystem management plan.
  - ▶ The LP optimizes the timberland operations for maximizing profitability while satisfying constraints including sustainability.
  - ▶ The model has around 8,500 functional constraints and 353,000 variables.
- ▶ The environment keeps **changing!**
  - ▶ E.g., climate, supply and demand, logging costs, and regulations.
  - ▶ Sensitivity analysis is applied.
- ▶ Read the application vignette in Section 6.7 and the article on CEIBA.

## “What-if” questions

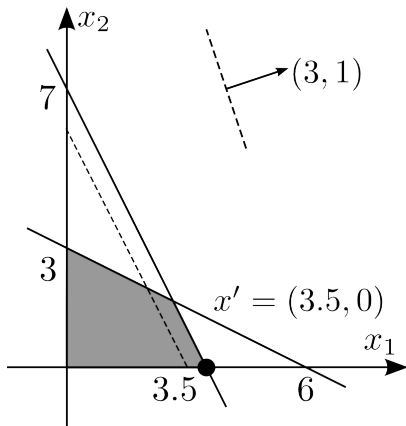
- ▶ In general, what-if questions can always be answered by formulating and solving a new optimization problem **from scratch**.
- ▶ But this may be too time consuming!
- ▶ By sensitivity analysis techniques:
  - ▶ The original optimal tableau provides useful information.
  - ▶ We typically start from the original optimal bfs and do **just a few** iterations to reach the new optimal bfs.
  - ▶ Duality provides a theoretical background.
- ▶ Here we want to introduce just one type of what-if question: What if I have **additional** units of a certain **resource**?
- ▶ Consider the following scenario:
  - ▶ One day, a salesperson enters your office and wants to offer you one additional unit of wood at \$1. Should you accept or reject?

## One more unit of wood

- ▶ To answer this question, you may formulate a new LP:

$$\begin{array}{lll} \max & 3x_1 & + \quad x_2 \\ \text{s.t.} & 2x_1 & + \quad x_2 \leq 7 \\ & x_1 & + \quad 2x_2 \leq 6 \\ & x_i & \geq 0 \quad \forall i = 1, 2. \end{array}$$

- ▶ The new objective value  $z' = 3 \times 3.5 = 10.5$  is larger than the old objective value  $z^* = 9$ .
- ▶ It is good to accept the offer (at the unit price \$1).
  - ▶ We earn \$0.5 as our **net benefit**.



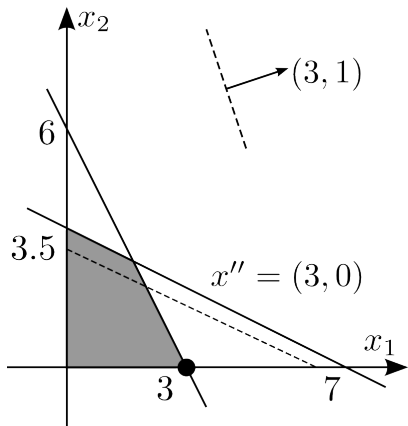


## One more labor hour

- ▶ Suppose instead of offering one additional unit of wood, the salesperson offers one additional labor hour at \$1.

$$\begin{array}{llll} \max & 3x_1 & + & x_2 \\ \text{s.t.} & 2x_1 & + & x_2 \leq 6 \\ & x_1 & + & 2x_2 \leq 7 \\ & x_i & \geq & 0 \quad \forall i = 1, 2. \end{array}$$

- ▶ The new objective value is **the same as** the old objective value.
- ▶ It is not worthwhile to buy it: The objective value does not increase.
  - ▶ The **net loss** is \$1.



## Shadow prices

- ▶ For each resource, there is a **maximum amount of price** we are willing to pay for one additional unit.
  - ▶ That depends on the net benefit of that one additional unit.
  - ▶ For wood, this price is \$1.5. For labor hours, this price is \$0.
- ▶ This motivates us to define **shadow prices** for each constraint:

### Definition 1 (Shadow price)

*For an LP that has an optimal solution, the shadow price of a constraint is the amount of objective value increased when the RHS of that constraint is increased by 1, assuming the current optimal basis remains optimal.*

- ▶ So for our table-chair example, the shadow prices for constraints 1 and 2 are 1.5 and 0, respectively.
- ▶ For shadow prices, see Section 4.7.
- ▶ Note that we **assume** that the current optimal basis does not change.

## Assuming the optimal basis does not change

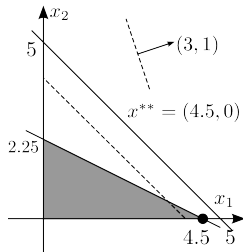
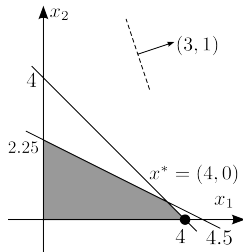
- ▶ Consider another example:

$$\begin{aligned} z^* = \max \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 4 \\ & x_1 + 2x_2 \leq 4.5 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

- ▶ If we want to find the shadow price of constraint 1, we may try to solve a new LP:

$$\begin{aligned} z^{**} = \max \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & x_1 + 2x_2 \leq 4.5 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

- ▶ Though  $z^{**} = 13.5$  and  $z^* = 12$ , the shadow price is  $15 - 12 = 3$ , **not** 1.5!
- ▶ Shadow prices measure the **rate** of improvement.



## Signs of shadow prices

- As a shadow price measures how the objective value is **increased**, its sign is determined based on how the feasible region changes:

### Proposition 7 (Signs of shadow prices)

For any LP, the sign of a shadow price follows the rule below:

<i>Objective function</i>	<i>Constraint</i>		
	$\leq$	$\geq$	$=$
max	$\geq 0$	$\leq 0$	Free
min	$\leq 0$	$\geq 0$	Free

## Nonbinding constraints' shadow prices

- ▶ If shifting a constraint does not affect the optimal solution, the shadow price must be **zero**.<sup>3</sup>

### Proposition 8

*Shadow prices are zero for constraints that are nonbinding at the optimal solution.*

- ▶ Now we know finding shadow prices allows us to answer the questions regarding additional units of resources.
- ▶ But how to find **all** shadow prices?
  - ▶ Let  $m$  be the number of constraints.
  - ▶ Is there a better way than solving  $m$  LPs?
  - ▶ Duality helps!

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<sup>3</sup>Not all binding constraints has nonzero shadow prices. Why?

## Dual optimal solution provide shadow prices

### Proposition 9

*For any LP, shadow prices equal the values of dual variables in the dual optimal solution.*

*Proof.* Let  $B$  be the old optimal basis and  $z = c_B^T A_B^{-1} b$  be the old objective value. If  $b_1$  becomes  $b'_1 = b_1 + 1$ , then  $z$  becomes

$$z' = c_B^T A_B^{-1} \left( b + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right) = z + (c_B^T A_B^{-1})_1.$$

So the shadow price of constraint 1 is  $(c_B^T A_B^{-1})_1$ . In general, the shadow price of constraint  $i$  is  $(c_B^T A_B^{-1})_i$ . As  $c_B^T A_B^{-1}$  is the dual optimal solution, the proof is complete. □

## An example

- ▶ What are the shadow prices?

$$\begin{array}{ll}
 \min & 6x_1 + 4x_2 \\
 \text{s.t.} & x_1 + x_2 \geq 2 \\
 & 3x_1 + x_2 \geq 1 \\
 & x_i \geq 0 \quad \forall i = 1, 2.
 \end{array}$$

- ▶ We solve the dual LP

$$\begin{array}{ll}
 \max & 2y_1 + y_2 \\
 \text{s.t.} & y_1 + 3y_2 \leq 6 \\
 & y_1 + y_2 \leq 4 \\
 & y_i \geq 0 \quad \forall i = 1, 2.
 \end{array}$$

The dual optimal solution is  $y^* = (4, 0)$ .

- ▶ So shadow prices are 4 and 0, respectively.

