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Suggested Solution for Pre-lecture Problems for Lecture 6

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1. The dual LP is

$$\begin{array}{llllllll}
\min & 10y_1 & + & 16y_2 & + & 14y_3 & & \\
\text{s.t.} & 2y_1 & & & + & y_3 & \geq & 4 \\
 & y_1 & + & y_2 & + & 3y_3 & \leq & -2 \\
 & & & y_2 & - & 3y_3 & = & 1 \\
 & \multicolumn{7}{l}{y_1 \geq 0, \quad y_2 \leq 0, \quad y_3 \quad urs.}
\end{array}$$

2. (a) Its standard form is

$$\begin{array}{llllllllll}
\min & 3x_1 & + & 5x_2 & & & & & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & & & = & 8 \\
 & x_1 & + & 2x_2 & & & + & x_4 & = & 12 \\
 & \multicolumn{9}{l}{x_i \geq 0 \quad \forall i = 1, 2.}
\end{array}$$

$$\begin{array}{cccc|c}
-3 & -5 & 0 & 0 & 0 \\ \hline
\boxed{1} & 1 & 1 & 0 & x_3 = 8 \\
1 & 2 & 0 & 1 & x_4 = 12
\end{array}
\rightarrow
\begin{array}{cccc|c}
0 & -2 & 3 & 0 & 24 \\ \hline
1 & 1 & 1 & 0 & x_1 = 8 \\
0 & \boxed{1} & -1 & 1 & x_4 = 4
\end{array}$$

$$\rightarrow
\begin{array}{cccc|c}
0 & 0 & 1 & 2 & 32 \\ \hline
1 & 0 & 2 & -1 & x_1 = 4 \\
0 & 1 & -1 & 1 & x_2 = 4
\end{array}$$

$$x^* = (4, 4, 0, 0).$$

(b) The dual LP is

$$\begin{array}{llllll}
\min & 8y_1 & + & 12x_2 & & \\
\text{s.t.} & y_1 & + & y_2 & \geq & 3 \\
 & y_1 & + & 2y_2 & \geq & 5 \\
 & \multicolumn{5}{l}{y_i \geq 0 \quad \forall i = 1, 2.}
\end{array}$$

(c) Its Phase-I LP of the dual is

$$\begin{array}{llllllllllllll}
\min & & & & & & & & & y_5 & + & y_6 & & \\
\text{s.t.} & y_1 & + & y_2 & - & y_3 & & & + & y_5 & & & = & 3 \\
 & y_1 & + & 2y_2 & & & - & y_4 & & & + & y_6 & = & 5 \\
 & \multicolumn{13}{l}{y_i \geq 0 \quad \forall i = 1, ..., 6.}
\end{array}$$

$$\begin{array}{cccccc|c}
0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline
1 & 1 & -1 & 0 & 1 & 0 & x_5 = 3 \\
1 & 2 & 0 & -1 & 0 & 1 & x_6 = 5
\end{array}
\overset{\text{adjust}}{\overbrace{\rightarrow}}
\begin{array}{cccccc|c}
2 & 3 & -1 & -1 & 0 & 0 & 8 \\ \hline
\boxed{1} & 1 & -1 & 0 & 1 & 0 & x_5 = 3 \\
1 & 2 & 0 & -1 & 0 & 1 & x_6 = 5
\end{array}$$

$$\rightarrow
\begin{array}{ccccc|c}
0 & 1 & 1 & -1 & 0 & 2 \\ \hline
1 & 1 & -1 & 0 & 0 & x_1 = 3 \\
0 & \boxed{1} & 1 & -1 & 1 & x_6 = 2
\end{array}
\rightarrow
\begin{array}{cccc|c}
0 & 0 & 0 & 0 & 0 \\ \hline
1 & 0 & -2 & 1 & x_1 = 1 \\
0 & 1 & 1 & -1 & x_2 = 2
\end{array}$$

And then the Phase-II

-8	-12	0	0	0
1	0	-2	1	$x_1 = 1$
0	1	1	-1	$x_2 = 2$

$$y^* = (1, 2, 0, 0).$$

$$c^T x^* = \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 32 = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \end{bmatrix} = (y^*)^T b.$$

3. (a)

$$A_B^{-1} b = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = x^*.$$

(b)

$$c_B^T A_B^{-1} = \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \end{bmatrix} = y^*.$$

(c) By proposition 9, shadow prices equal the values of dual variables in the dual optimal solution. Therefore, the shadow price for the first and the second primal constraints are 1 and 2, respectively.