Operations Research, Spring 2016
Suggested Solution for Pre-lecture Problems for Lecture 10

Solution providers: Kiwi Liu and Johnny Chen
Department of Information Management
National Taiwan University

1. (a) G is shown in Figure 1.

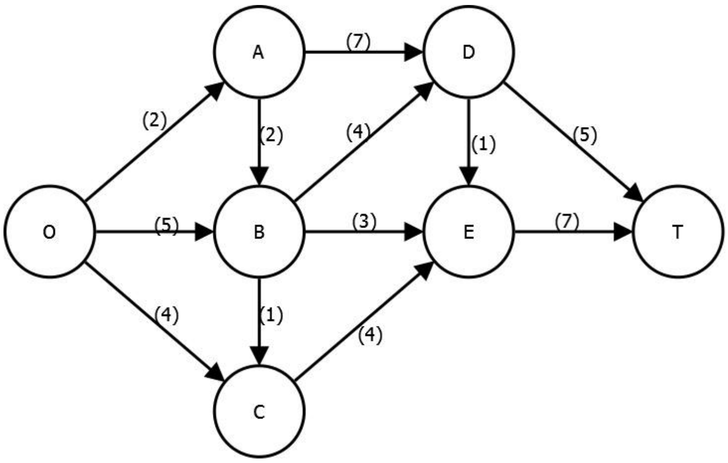

Figure 1: Graphical solution for Problem 1.(a)

(b) Let the parameters be

$$\begin{aligned} V &= \text{the set of vertices} = \{O, A, B, C, D, E, T\}, \\ E &= \text{the set of arcs} = \{(O, A), (O, B), (O, C), (A, B), (A, D), (B, C), \\ &\qquad (B, D), (B, E), (C, E), (D, E), (D, T), (E, T)\}, \\ D_{ij} &= \text{the distances of arc } (i, j),\ (i, j) \in E. \end{aligned}$$

Let the decision variables be

$$x_{ij} = \begin{cases} 1 & \text{if the arc } (i,j) \text{ is chosen} \\ 0 & \text{otherwise} \end{cases}, (i, j) \in E.$$

The integer program is

$$\begin{aligned} \min \quad & \sum_{(i,j)\in E} D_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{(O,j)\in E} x_{Oj} = 1 \\ & \sum_{(i,T)\in E} x_{iT} = 1 \\ & \sum_{(i,k)\in E} x_{ik} - \sum_{(k,j)\in E} x_{kj} = 0 \quad \forall k \in V \\ & x_{ij} \in \{0, 1\} \quad \forall (i, j) \in E. \end{aligned}$$

(c) We rewrite the model as follow:

$$
\begin{array}{lllllllllllllll}
\min & \displaystyle\sum_{(i,j)\in E} D_{ij}x_{ij} \\
\text{s.t.} & x_{OA}+x_{OB}+x_{OC} & & & & & & & & & & & & & =1 \\
& x_{OA} & & & -x_{AB}-x_{AD} & & & & & & & & & & =0 \\
& & x_{OB} & & +x_{AB} & & -x_{BC}-x_{BD}-x_{BE} & & & & & & & & =0 \\
& & & x_{OC} & & & +x_{BC} & & & -x_{CE} & & & & & =0 \\
& & & & x_{AD} & & +x_{BD} & & & & -x_{DT}-x_{DE} & & & & =0 \\
& & & & & & & x_{BE}+x_{CE} & & & +x_{DE}-x_{ET} & & & & =0 \\
& & & & & & & & & & x_{DT} & +x_{ET} & & & =1
\end{array}
$$

$$x_{ij}\in\{0,1\} \quad \forall (i,j)\in E.$$

Then, we get

$$
A=\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

By Proposition 2 in lecture 10 (p.17), matrix A satisfies the three conditions.

i. All elements in A are either 1, 0 or -1.

ii. Each column indeed contains at most two nonzero elements.

iii. For the matrix A in this problem, we may let group 1 contains row 1 and group 2 contains the other rows. Then for each column, two nonzero elements are either (1) in the same group (which must be group 2) and different or (2) in different groups and the same.

Note. A counter example might be

$$A''=\begin{bmatrix} 1 & 1\\ 1 & -1\end{bmatrix}$$

Then, we can find that the first row may be group of 1, but the second row cannot be the same group. Thus, in this counter example, it does not satisfied (iii).

As a result, we know the coefficient matrix of IP in this problem is totally unimodular.

2. (a) Let the parameters be

$$
\begin{aligned}
V &= \text{the set of vertices} = \{O,A,B,C,D,E,T\},\\
E &= \text{the set of arcs} = \{(O,A),(O,B),(O,C),(A,B),(A,D),(B,C),\\
&\qquad (B,D),(B,E),(C,E),(D,E),(D,T),(E,T)\},\\
C_{ij} &= \text{the capacities of arc } (i,j),\ (i,j)\in E.
\end{aligned}
$$

Let the decision variables be

$$x_{ij} = \text{the flow size of the arc (i, j).}$$

We add an artificial arc (T,O). Let x_{TO} be the flow size, $C_{TO}=-1$ be unit cost of the added arc (T,O).

The integer program is

$$
\begin{aligned}
\min \quad & -x_{TO}\\
& \sum_{(i,k)\in E} x_{ik} - \sum_{(k,j)\in E} x_{kj} = 0 \quad \forall k\in V\\
& x_{ij}\le C_{ij} \quad \forall (i,j)\in E\\
& x_{ij}\in\mathbb{Z}_+ \quad \forall (i,j)\in E.
\end{aligned}
$$

(b) We rewrite the model in (a) as follow:

$$
\begin{array}{llllllllllllll}
\min & -x_{TO} \\
\text{s.t.} & x_{OA} & & & -x_{AB} - x_{AD} & & & & & & & = 0 \\
& & x_{OB} & & +x_{AB} & -x_{BC} - x_{BD} - x_{BE} & & & & & & = 0 \\
& & & x_{OC} & & +x_{BC} & & -x_{CE} & & & & = 0 \\
& & & & x_{AD} & +x_{BD} & & & -x_{DT} - x_{DE} & & & = 0 \\
& & & & & & x_{BE} + x_{CE} & & +x_{DE} - x_{ET} & & & = 0 \\
& & & & & & & & x_{DT} & +x_{ET} & & = 0
\end{array}
$$

$$
\begin{aligned}
& x_{ij} \leq C_{ij} \quad \forall (i,j) \in E \\
& x_{ij} \in \mathbb{Z}_+ \quad \forall (i,j) \in E.
\end{aligned}
$$

Then, we get

$$
A = \begin{bmatrix}
1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

We can prove by the same reason as 1c, the matrix A also satisfies the Proposition 2. Notice that we do not need to consider the capacity constraints, as indicated in the slides. As a result, the coefficient matrix of maximum flow problem is totally unimodular.