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Suggested Solution For Homework 2

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1. (a) Let the decision variables be

$$\begin{aligned} p_1 &= \text{number of product 1 produced in a day,} \\ p_2 &= \text{number of product 2 produced in a day,} \\ p_3 &= \text{number of product 3 produced in a day,} \end{aligned}$$

$$\begin{aligned} \max \quad & 10p_1 + 15p_2 + 25p_3 \\ \text{s.t.} \quad & 2p_1 + 3p_2 + 5p_3 \leq 300 \\ & 3p_1 + 6p_2 + 7p_3 \leq 400 \\ & 4p_1 + 3p_2 + 9p_3 \leq 500 \\ & p_2 + 10p_3 \leq 600 \\ & p_1 \leq 100 \\ & p_2 \leq 80 \\ & p_3 \leq 50 \\ & p_i \geq 0 \quad \forall i = 1, 2, 3. \end{aligned}$$

(b) The dual program is :

$$\begin{aligned} \min \quad & 300y_1 + 400y_2 + 500y_3 + 600y_4 + 100y_5 + 80y_6 + 50y_7 \\ \text{s.t.} \quad & 2y_1 + 3y_2 + 4y_3 + y_5 \geq 10 \\ & 3y_1 + 6y_2 + 3y_3 + y_4 + y_6 \geq 15 \\ & 5y_1 + 7y_2 + 9y_3 + 10y_4 + y_7 \geq 25 \\ & y_i \geq 0 \quad \forall i = 1, ..., 7. \end{aligned}$$

(c) According to the primal optimal solution, the number of material 1 be used is 280, material 2 be used is 400, material 3 be used is 500, material 4 be used is $503\frac{1}{3}$.

y_1, y_4, y_5 and y_6 will be 0 because the primal problem's corresponding constraints are not binding.

Constraints 1,2 and 3 in dual problem will be binding because p_1, p_2 and p_3 are not 0.

(d) According to 1.(c) we know $y_1 = y_4 = y_5 = y_6 = 0$ and constraints 1, 2, and 3 are binding. Thus, the formulation become :

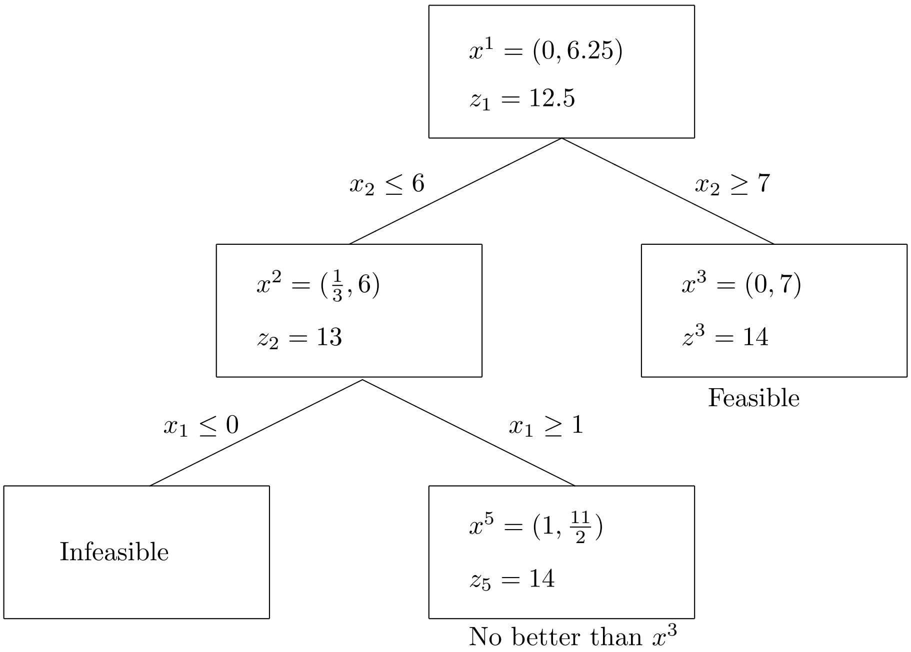
$$\begin{aligned} \min \quad & 400y_2 + 500y_3 + 50y_7 \\ \text{s.t.} \quad & 3y_2 + 4y_3 = 10 \\ & 6y_2 + 3y_3 = 15 \\ & 7y_2 + 9y_3 + y_7 = 25 \\ & y_i \geq 0 \quad \forall i = 1, ..., 7. \end{aligned}$$

The optimal solution from y_1 to y_7 is (0, 2, 1, 0, 0, 0, 2) , and the optimal objective value is 1400 for both primal and dual problem.

(e) The shadow price of primal problem's constraint 2 is 2 so you should buy material 2 at \$1.5 per unit.

The shadow price of primal problem's constraint 3 is 1 so you should not buy material 3 at \$1.5 per unit.

2. (a) The optimal solution is x^3 =(0, 7) with optimal value $z^* = 14$.



(b)

$$\begin{aligned} \max \quad & 10y_1 + 25y_2 \\ \text{s.t.} \quad & y_1 + 3y_2 \leq 3 \\ & 2y_1 + 4y_2 \leq 2 \\ & y_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

(c) Solve the dual problem by simplex method then we get $y_1 = 0$ and $y_2 = 0.5$, so the shadow price of constraint 1 is 0 and the shadow price of constraint 2 is 0.5.
$y_1 = 0$, $y_2 = 0.5$, and $z^{**} = 12.5$.

$$\begin{array}{cccc|c} -10 & -25 & 0 & 0 & 0 \\ \hline 1 & 3 & 1 & 0 & y_3 = 3 \\ 2 & 4 & 0 & 1 & y_4 = 2 \end{array}$$

$$\rightarrow \begin{array}{cccc|c} \frac{5}{2} & 0 & 0 & \frac{25}{4} & \frac{25}{2} \\ \hline -\frac{1}{2} & 0 & 1 & -\frac{3}{4} & y_3 = \frac{3}{2} \\ \frac{1}{2} & 1 & 0 & \frac{1}{4} & y_2 = \frac{1}{2} \end{array}$$

3. (a) The node with the highest objective value may have descendants with higher objective value of feasible solution. Then you can omit the other nodes which has lower objective value.

(b) You may find the integer solution faster.

4. (a) Let the decision variables be

$$x_{ij} = \begin{cases} 1 & \text{if item } i \text{ is chosen and put in bag } j \\ 0 & \text{otherwise.} \end{cases}, i = 1, ..., 12, j = 1, 2.$$

Let the parameters be

$$I_i = \text{importance of item } i, i = 1, ..., 12,$$
$$W_i = \text{weight of item } i., i = 1, ..., 12.$$

The IP formulation is

$$\begin{aligned}
\max \quad & \sum_{j=1}^{2}\sum_{i=1}^{12} I_i x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{12} W_i x_{ij} \le 20 \quad \forall j = 1, 2 \\
& \sum_{j=1}^{2} x_{ij} \le 1 \quad \forall i = 1, ..., 12 \\
& x_{ij} \in \{0, 1\} \quad \forall i = 1, ..., 12 \quad \forall j = 1, 2.
\end{aligned}$$

(b) Let the decision variables be

$$x_{ij} = \begin{cases} 1 & \text{if item } i \text{ is chosen and put in bag } j \\ 0 & \text{otherwise.} \end{cases}, i = 1, ..., 12, j = 1, 2.$$

Let the parameters be

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\max \quad & \sum_{j=1}^{2}\sum_{i=1}^{12} I_i x_{ij} \\
\text{s.t.} \quad & \sum_{i=1}^{12} W_i x_{ij} \le 20 \quad \forall j = 1, 2 \\
& \sum_{j=1}^{2} x_{ij} \le 1 \quad \forall i = 1, ..., 12 \\
& x_{2j} + x_{3j} \le 1 \quad \forall j = 1, 2 \\
& x_{4j} + x_{5j} + x_{6j} \le 2 \quad \forall j = 1, 2 \\
& \sum_{j=1}^{2} (x_{8j} + x_{9j} + x_{10,j} + x_{11,j} + x_{12,j}) \ge 2 \\
& \sum_{j=1}^{2} (x_{1j} + x_{2j}) \ge 1 - \sum_{j=1}^{2} x_{3j} \\
& x_{ij} \in \{0, 1\} \quad \forall i = 1, ..., 12 \quad \forall j = 1, 2.
\end{aligned}$$

(c) Let the decision variables be

$$x_{ij} = \begin{cases} 1 & \text{if item } i \text{ is chosen and put in bag } j \\ 0 & \text{otherwise.} \end{cases}, i = 1, ..., 12, j = 1, 2.$$

Let the parameters be

$$I_i = \text{importance of item } i, i = 1, ..., 12,$$
$$W_i = \text{weight of item } i., i = 1, ..., 12.$$

The IP formulation is

$$\begin{aligned}
\min \quad & k \\
\text{s.t.} \quad & k \geq \sum_{i=1}^{12} (x_{i1} - x_{i2}) \\
& k \geq \sum_{i=1}^{12} (x_{i2} - x_{i1}) \\
& \sum_{j=1}^{2} \sum_{i=1}^{12} I_i x_{ij} = z^* \\
& \sum_{i=1}^{12} W_i x_{ij} \leq 20 \quad \forall j = 1, 2 \\
& \sum_{j=1}^{2} x_{ij} \leq 1 \quad \forall i = 1, ..., 12 \\
& x_{ij} \in \{0, 1\} \quad \forall i = 1, ..., 12 \quad \forall j = 1, 2.
\end{aligned}$$

5. Let the decision variables be

$$x_i = \begin{cases} 1 & \text{if an ambulance is located at district } i \\ 0 & \text{otherwise.} \end{cases}, i = 1, ..., 8, j = 1, .., 8,$$

$$y_{ij} = \begin{cases} 1 & \text{if an ambulance is located at district } i \text{ and is sent to district } j \\ 0 & \text{otherwise.} \end{cases}, i = 1, ..., 8, j = 1, .., 8.$$

Let the parameters be

$$\begin{aligned}
T_{ij} &= \text{traveling time between district } i \text{ and } j, i = 1, ..., 8, j = 1, .., 8, \\
P_j &= \text{population in district } j, j = 1, .., 8.
\end{aligned}$$

The IP formulation is

$$\begin{aligned}
\min \quad & M \\
\text{s.t.} \quad & M \geq y_{ij} T_{ij} P_j \\
& y_{ij} \leq x_i \quad \forall i = 1, ..., 8 \quad \forall j = 1, ..., 8 \\
& \sum_{i=1}^{8} x_i = 2 \\
& \sum_{i=1}^{8} y_{ij} = 1 \quad \forall j = 1, ..., 8 \\
& x_i, y_{ij} \in \{0, 1\} \quad \forall i = 1, ..., 8 \quad \forall j = 1, ..., 8.
\end{aligned}$$

6. (a) The model of problem 4.b is

```
param I; #12
param B; #2

param Importance{i in 1..I};
param Weight{i in 1..I};

var x{i in 1..I, j in 1..B} binary;
```

```
maximize importance:
sum{j in 1..B}sum{i in 1..I}(Importance[i]*x[i,j]);


subject to WeightConstraint{j in 1..B}: #one bag can only afford 20 kilograms.
sum{i in 1..I}(Weight[i]*x[i, j])<=20 ;
subject to ItemConstraint{i in 1..I}: #one item can only be put in one bag.
sum{j in 1..B}(x[i, j])<=1 ;
subject to ChooseConstraint1{j in 1..B}: #items 2 and 3 cannot be put in the same bag.
x[2,j]+x[3,j]<=1 ;
subject to ChooseConstraint2{j in 1..B}: #items 4, 5 and 6 cannot be put in the same bag.
x[4,j]+x[5,j]+x[6,j]<= 2 ;
subject to ChooseConstraint3: #at least two of items 8 to 12 must be carried.
sum{j in 1..B}(x[8,j]+x[9,j]+x[10,j]+x[11,j]+x[12,j])>=2 ;
subject to ChooseConstraint4: #at least one of items 1 and 2 must be carried if item 3 is n
sum{j in 1..B}(x[1,j]+x[2,j])>=1-(sum{j in 1..B}(x[3,j])) ;
```

The optimal objective value of 4.b is 44, and the value of x are shown in Table 1. We should put item 1, 2, 5, 8 in bag 1, item 4, 9, 10, 11, 12 in bag 2. Moreover, there are three answers which are also correct. They are $\{(1, 2, 4, 5)(8, 9, 10, 11, 12)\}$, $\{(1, 2, 4, 9)(5, 8, 10, 11, 12)\}$ and $\{(1, 7, 8)(4, 9, 10, 11, 12)\}$.

	$x[*][1]$	$x[*][2]$
1	1	0
2	1	0
3	0	0
4	0	1
5	1	0
6	0	0
7	0	0
8	1	0
9	0	1
10	0	1
11	0	1
12	0	1

Table 1: Solution of our model

(b) The model of problem 5 is

```
param L; #8

param Time{i in 1..L, j in 1..L};
param Population{j in 1..L};

var x{i in 1..L} binary; #whether district i has an ambulance or not
var y{i in 1..L, j in 1..L} binary; #whether an ambulance can be sent from district i to j
var T; #maximum population-weighted firefighting time

minimize cost:
T;
```

```
subject to Constraint{i in 1..L, j in 1..L}:
T >= y[i,j]*Time[i,j]*Population[j];
subject to AvailableConstraint{i in 1..L, j in 1..L}:
y[i,j]<=x[i];
subject to ItemConstraint: #There are only two anbulances.
sum{i in 1..L}(x[i])=2 ;
subject to ChooseConstraint1{j in 1..L}: #One district can only be served by one ambulance.
sum{i in 1..L}(y[i,j])=1 ;
```

The optimal objective value of 5 is 135. We should locate the ambulances at district 1 and 6, and the separation plan is shown in Table 2. We should send an ambulance from district 1 to district 1, 2, 4 and send send an ambulance from district 6 to district 3, 5, 6, 7 and 8. Moreover, following answers are correct. They are {(1, 2, 5)(3, 4, 6, 7, 8)} and {(1, 2)(3, 4, 5, 6, 7, 8)}.

	to 1	to 2	to 3	to 4	to 5	to 6	to 7	to 8
from 1	1	1	0	1	0	0	0	0
from 2	0	0	0	0	0	0	0	0
from 3	0	0	0	0	0	0	0	0
from 4	0	0	0	0	0	0	0	0
from 5	0	0	0	0	0	0	0	0
from 6	0	0	1	0	1	1	1	1
from 7	0	0	0	0	0	0	0	0
from 8	0	0	0	0	0	0	0	0

Table 2: Separation plan