Operations Research, Spring 2017
Homework 3

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1 Problems

1. (20 points; 5 points each) A retailer sells a product to a market. If the price is p, the demand quantity is ae^{-bp}, where $a > 0$ and $b > 0$ are fixed constants. Let $c > 0$ be the unit procurement cost, the retailer's profit-maximizing problem is

$$\max_p \ \pi(p) = ae^{-bp}(p - c).$$

   (a) Find the gradient and Hessian of $\pi(p)$.
   Hint. While gradients and Hessian matrices are introduced for multi-variate NLPs, they also apply to single-variate NLPs!

   (b) When is the Hessian negative semi-definite? Under what condition is $\pi(p)$ concave over $\mathbb{R}$?
   Note. The condition has p inside.

   (c) Find a price $\bar{p}$ as a function of a, b, and c that satisfies the first-order condition solution of $\pi(p)$. Is the Hessian negative semi-definite at $\bar{p}$? May we conclude that $\bar{p}$ is a global optimal solution? Explain why.

   (d) For $p \in [c, \infty)$, show that the gradient of $\pi(p)$ is first positive and then negative. Combining this finding with what you have in Part (c), may we conclude that $\bar{p}$ is a global optimal solution? Explain why.

2. (20 points; 5 points each) A retailer sells products 1 and 2 at supply quantities q_1 and q_2. For product i, the market-clearing price is

$$p_i = a_i - b_i q_i, \quad i = 1, 2,$$

   where $a_i > 0$ and $b_i > 0$ for $i = 1, 2$. The retailer sets q_1 and q_2 to maximize its total profit while ensuring that the total supply does not exceed $K > 0$.

   (a) Formulate the retailer's problem. Is this a convex program?

   (b) Write down the complete KKT condition for the problem, including primal feasibility, dual feasibility, and complementary slackness.

   (c) Solve this problem by finding an optimal analytical solution.

   (d) How do the optimal quantities change with K? Mathematically show it and intuitively explain why.

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3. (20 points; 5 points) Consider the nonlinear program

$$\min_{x \in \mathbb{R}^2} \ f(x_1, x_2) = 2e^{x_1} + \frac{x_2^2}{2}.$$

   (a) Find the gradient and Hessian of f.

   (b) Starting at $(2, 2)$, run one iteration of Newton's method to get to the next solution.

   (c) Starting at $(2, 2)$, run one iteration of the gradient descent method to get to the next solution. Choose the step size to be $a = 1$.

   (d) Starting at $(2, 2)$, run one iteration of the gradient descent method to get to the next solution. Choose the step size to be the one that reaches the lowest point along the improving direction. **Hint.** Is this problem unbounded?

4. (20 points) You are invited to design a formulation problem and provide its answer. The problem should be new, i.e., not too similar to any problem taught in this course. The problem should be doable, i.e., a solution does exist and may be found by most students in this course. The problem should be challenging, i.e., cannot be solved trivially. Ideally, an around-average student in this course should be able to understand the problem and formulate a right model in 15 to 25 minutes. Hopefully the problem is creative and interesting.

   The problem statement can be written in English or Chinese. The answer should be a complete formulation as a linear program, a linear integer program, or a convex program. It cannot be a nonlinear integer program; it cannot be a nonconvex program.

   As long as your program description is precise (i.e., of no ambiguity) and your answer is correct, you get 15 points. The remaining 5 points depend on the novelty, appropriateness (not too easy or too hard), creativity, and level of interesting.

5. (20 points) Write down the full English names of the TAs and instructor to get 20 points. One name counts for 5 points.

2 Submission rules

The deadline of this homework is 2 pm, May 15, 2017. Please put a hard copy of the work into the instructor's mailbox on the first floor of the Management Building 2 by the due time. Works submitted between 2 pm and 3 pm will get 10 points deducted as a penalty. Submissions later than 3 pm will not be accepted. Each student must submit her/his individual work.