

Operations Research, Spring 2017
Lecture 5: LP Duality

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1. (3 points) Find the dual for the following LP:

$$\begin{aligned} \max \quad & 4x_1 - x_2 + 2x_3 \\ \text{s.t.} \quad & x_1 + x_2 \leq 5 \\ & 2x_1 + x_2 \leq 7 \\ & 2x_2 + x_3 \geq 6 \\ & x_1 + x_3 = 4 \\ & x_1 \geq 0, x_2 \text{ urs}, x_3 \text{ urs.} \end{aligned}$$

2. Consider a primal LP

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 16 \\ & x_2 \leq 8 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

- (a) (2 points) Solve this LP and show that it is unbounded.
- (b) (2 points) Formulate and solve the dual LP.
- (c) (2 points) How would you explain the primal unboundedness and dual infeasibility? Is it a coincidence?

3. Consider a primal LP

$$\begin{aligned} \max \quad & 3x_1 + 7x_2 + 5x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leq 50 \\ & 2x_1 + 3x_2 + x_3 \leq 100 \\ & x_i \geq 0 \quad \forall i = 1, \dots, 3. \end{aligned}$$

- (a) (3 points) Use the simplex method to find an optimal tableau.
- (b) (2 points) What is the primal optimal solution associated with this optimal tableau? Is it unique?
- (c) (3 points) Formulate the dual program, solve it graphically, and show that the values of dual variables in the dual optimal solution are indeed the primal shadow prices.
- (d) (3 points) Without formulating or solving the dual program, find the shadow prices for both constraints.

4. IEDO Glass manufactures glasses for wine, beer, champagne, and whiskey. Each type of glass requires time in the molding shop, time in the packaging shop, and a certain amount of glass. The resources required to make each type of glass are given below. The last column records the available amount of each resource.

	Wine	Beer	Champagne	Whiskey	Capacity
Molding time (minutes)	4	9	7	10	600
Packaging time (minutes)	1	1	3	40	400
Glass (oz)	3	4	2	1	500
Selling price	\$6	\$10	\$9	\$20	—

Let x_1 , x_2 , x_3 , and x_4 be the production quantities of wine, beer, champagne, and whiskey, respectively. An optimal solution is

$$(x_1^*, x_2^*, x_3^*, x_4^*) = \left(\frac{400}{3}, 0, 0, \frac{20}{3} \right).$$

(a) (3 points) Formulate an LP that maximizes IEDO Glass' sales revenue.

	Wine	Beer	Champagne	Whiskey	Capacity
Molding time (minutes)	4	9	7	10	600
Packaging time (minutes)	1	1	3	40	400
Glass (oz)	3	4	2	1	500
Selling price	\$6	\$10	\$9	\$20	–

$$(x_1^*, x_2^*, x_3^*, x_4^*) = \left(\frac{400}{3}, 0, 0, \frac{20}{3} \right).$$

- (a) (2 points) Let y_i be the dual variable corresponding to primal constraint i , $i = 1, 2, 3$. Find the dual LP. Without solving the dual LP, determine the objective value associated with any dual optimal solution.
- (b) (3 points) Without solving the dual LP, use the given primal optimal solution to show that $y_3 = 0$ at any dual optimal solution. Is there any economic interpretation for this result?

5. Complementary slackness can be used to solve an LP given an optimal solution to its dual LP. Consider the primal LP maximizing IEDO Glass' sales revenue:

	Wine	Beer	Champagne	Whiskey	Capacity
Molding time (minutes)	4	9	7	10	600
Packaging time (minutes)	1	1	3	40	400
Glass (oz)	3	4	2	1	500
Selling price	\$6	\$10	\$9	\$20	–

$$(x_1^*, x_2^*, x_3^*, x_4^*) = \left(\frac{400}{3}, 0, 0, \frac{20}{3} \right).$$

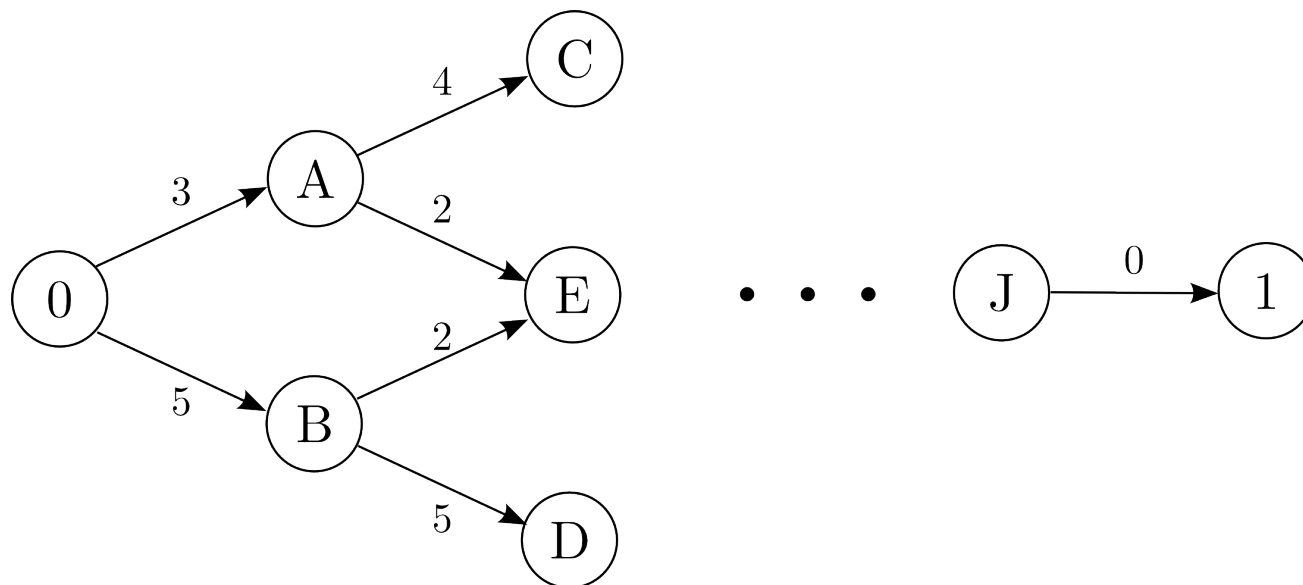
- (a) (3 points) For the dual LP, add slack variables to the functional constraints. With $x_1^* > 0$ and $x_4^* > 0$, comment on the first and fourth dual slack variables.
- (b) (3 points) With the fact that the third functional constraint is binding, we know $y_3^* = 0$ if it is the optimal value of the dual variable corresponding to the third primal constraint. Now, for the four dual equalities, three variables have been eliminated. May you find a dual optimal solution now?

6. Once a primal LP is solved, the corresponding dual LP is also “solved.” In particular, a dual optimal solution can be found by calculating $c_B^T A_B^{-1}$. It is therefore not surprising that once one uses AMPL to solve a primal LP, she/he can also use AMPL to obtain a dual optimal solution (and thus a set of shadow prices) in one shot.
- (a) (0 points) Use AMPL to solve a whatever LP you have.
 - (b) (0 points) Use the **display** statement to display a constraint name. Verify that these are shadow prices!

7. (Project management: introduction) To complete a project, ten tasks, each with a processing time, must be done. Tasks may be processed in parallel. However, a task may be started only after the completion of some other tasks (i.e., a task may have some “prerequisites”). The following table shows the processing times and prerequisites, if any, of each job.

Task	A	B	C	D	E	F	G	H	I	J
Processing time	3	5	4	5	2	6	8	5	4	6
Prerequisites	\emptyset	\emptyset	$\{A\}$	$\{B\}$	$\{A, B\}$	$\{C, E\}$	$\{D\}$	$\{F\}$	$\{G, H\}$	$\{I\}$

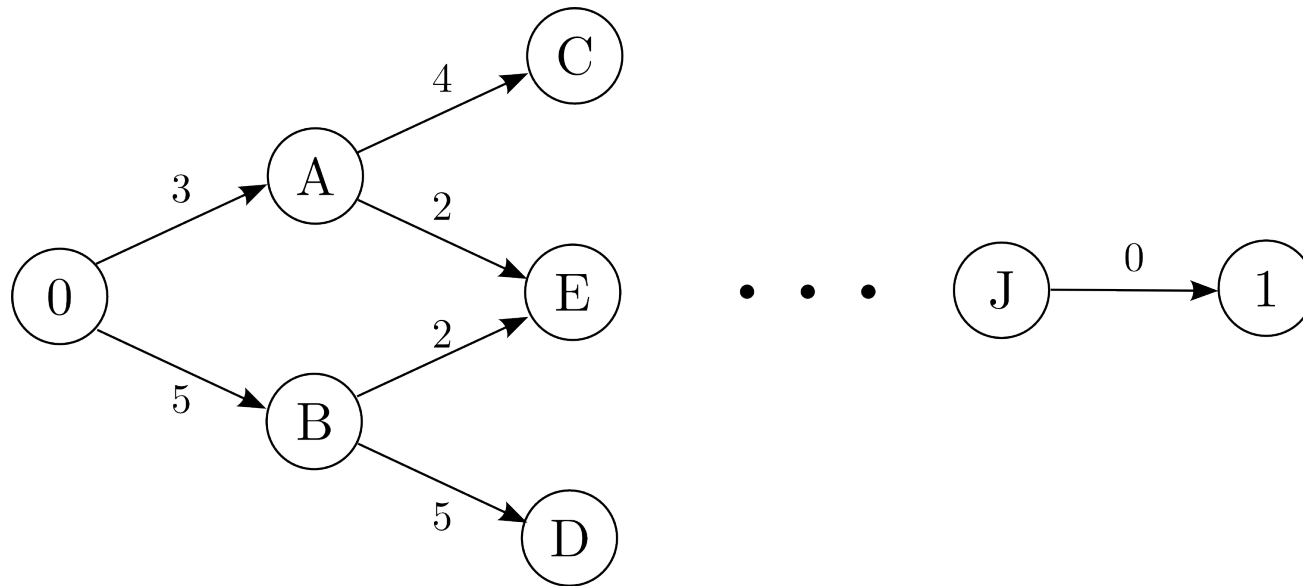
Given these ten tasks, a natural question is: How long does it take to complete the whole project if all tasks are started as soon as possible? To answer this problem, let’s start by visualizing the relationship among tasks:



In this figure, node i represents task i while c_{ij} of arc (i, j) represents the processing time of task j . If we complete the figure and find a *longest path* from node 0 (which means “starting”) to node 1 (which means “ending”), the length of the path will tell us the desired answer.

- (a) (3 points) Complete this figure and find a longest path.
- (b) (2 points) Find the starting and completion times of tasks A, E, and J.

Task	A	B	C	D	E	F	G	H	I	J
Processing time	3	5	4	5	2	6	8	5	4	6
Prerequisites	\emptyset	\emptyset	$\{A\}$	$\{B\}$	$\{A, B\}$	$\{C, E\}$	$\{D\}$	$\{F\}$	$\{G, H\}$	$\{I\}$



8. (Project Management: LP formulation) When the number of tasks and precedence relations are large enough, we need the help of mathematical programming for solving the problem. Let

x_i = the completion time of task i , $i = 0, A, B, \dots, I, 1$,

a constraint that relates tasks 0 and A is $x_A = x_0 + 3$. Similarly, a constraint that relates tasks A and C is $x_C = x_A + 4$. But how about tasks A, B, and E?

- (a) (3 points) Write down two constraints that relate tasks A, B, and E.
- (b) (3 points) Write down all constraints that relate tasks 0, A, ..., and E.
- (c) (2 points) Write down a constraint to set up x_0 and the objective function.
- (d) (3 points) Write down the compact formulation of this problem in general.

9. (Project management: critical path) Knowing the completion times are good, but we also want to identify the “bottlenecks” of the project: those tasks on the longest path (called the critical path in project management). These tasks are called bottlenecks because the project can be finished earlier if the processing time of a bottleneck task is shortened. Given all the values of completion times, of course there is a (somewhat tedious) way to find the bottlenecks. Here let’s apply LP duality and shadow prices.
- (a) (2 points) In our LP formulation, each arc has a constraint and thus a shadow price. Predict what are the shadow prices of arcs not on critical paths.
 - (b) (2 points) Predict what are the shadow prices of arcs on critical paths.
 - (c) (2 points) If we solve the dual LP, how may we find bottleneck tasks?