Operations Research, Spring 2017
Lecture 13: Algorithms for Nonlinear Programming

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1. (2 points each) Consider the NLP

$$\min_{x \in \mathbb{R}} \ f(x) = x^2 - 3x + 2.$$

(a) Find its gradient and Hessian.

(b) Starting at an initial solution $x^0 = 4$, do one iteration of gradient descent to reach the next solution.

(c) When applying gradient descent, is it always true to reach a global optimal solution if the function is convex? Explain why.

2. (2 points each) Consider the NLP

$$\min_{x \in \mathbb{R}^2} \; f(x) = x_1^2 - 2x_1x_2 + 2x_2^2 + 2x_1.$$

(a) Find its gradient and Hessian. Analytically find its optimal solution x^*.

(b) Starting at an initial solution $x^0 = (0, 0)$, do one iteration of gradient descent to reach the next solution x^1.

(c) Starting at the current solution x^1, do one iteration of gradient descent to reach the next solution x^2.

(d) Depict the search route from x^0 to x^1 to x^2. Also depict the optimal solution x^* and the two equations obtained from the FOC. What do you observe? In particular, would you predict x^3, x^4, etc.?

3. (2 points each) For a maximization problem, we apply *gradient ascend* rather than gradient descent: We move along the gradient, not the negative gradient. Consider the NLP

$$\max_{x \in \mathbb{R}^2} f(x) = x_1 x_2.$$

(a) Find its gradient and Hessian. Show that this problem is unbounded.

(b) Starting at an initial solution $x^0 = (1, -1)$, do one iteration of gradient descent to reach the next solution x^1. What will be the conclusion of running gradient descent?

(c) Starting at an initial solution $x^0 = (1, 1)$, do one iteration of gradient descent to reach the next solution x^1. What will be the conclusion of running gradient descent?

4. (2 points each) Consider the NLP

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq -5. \end{aligned}$$

Note that this is a constrained maximization problem.

(a) Analytically find its optimal solution.

(b) For a constrained problem, one simple idea to search for the step size is to search only among feasible step sizes. Following this idea, starting from an initial solution $x^0 = (-2, -4)$ and do one iteration of constrained gradient descent to reach the next solution x^1.

(c) Starting from an initial solution $x^0 = (-3, -3)$ and do one iteration of constrained gradient descent to reach the next solution x^1.

(d) Comment on this algorithm for constrained NLPs.

5. (2 points each) Consider the NLP

$$\min_{x \in \mathbb{R}} \; f(x) = (x-1)(x-2)(x+1)x + 2x.$$

(a) Start from an initial solution $x^0 = 2$, run one iteration of Newton's method to reach the next solution x^1. Show that you get improved, i.e., $f(x^1) < f(x^0)$.

(b) Start from an initial solution $x^0 = \frac{1}{2}$, run one iteration of Newton's method to reach the next solution x^1. Do you get improved?

(c) Start from an initial solution $x^0 = 0$, run one iteration of Newton's method to reach the next solution x^1. What happened?

6. (2 points each) Recall our retailer selling two products by choosing their prices p_1 and p_2. Suppose that the demand of product i is

$$q_i = 4 - p_i + \frac{1}{2}p_{3-i}, \quad i = 1, 2.$$

The unit production cost is c_i for product i.

(a) Formulate the retailer's problem.

(b) Suppose that $c_1 = c_2 = 2$. Starting from $x^0 = (0, 0)$, run one iteration of gradient descent to reach the next solution x^1.

(c) Suppose that $c_1 = c_2 = 2$. Starting from $x^0 = (0, 0)$, run one iteration of Newton's method to reach the next solution x^1.

(d) Suppose that $c_1 = 1$ and $c_2 = 2$. Starting from $x^0 = (0, 0)$, run one iteration of gradient descent to reach the next solution x^1.

(e) Suppose that $c_1 = 1$ and $c_2 = 2$. Starting from $x^0 = (0, 0)$, run one iteration of Newton's method to reach the next solution x^1.