Programming Design

Algorithms and Recursion

Ling-Chieh Kung

Department of Information Management
National Taiwan University

Outline

- **Algorithms and complexity**
- Recursion
- Searching and sorting

Introduction

- It is said that:
  - **Programming** = **Data structures** + **Algorithms**.
  - http://en.wikipedia.org/wiki/Algorithms_%2B_Data_Structures_%3D_Programs
  - To design a program, choose data structures to store your data and choose algorithms to process your data.
- Each of "data structures" and "algorithms" requires one (or more) courses.
  - We will only give you very basic ideas.

Algorithms

- Today we talk about **algorithms**, collections of steps for completing a task.
  - In general, an algorithm is used to **solve a problem**.
  - The most common strategy is to divide a problem into small pieces and then solve those **subproblems**.
  - We will introduce **recursion**, a way to solve a problem based on the solution/outcome of subproblems.
- For a problem, there may be multiple algorithms.
  - The first criterion, of course, is **correctness**.
  - **Time complexity** is typically the next for judging correct algorithms.
- As examples, we introduce two specific problems: **searching** and **sorting**.

Example: listing all prime numbers

- Given an integer n, let's list all the **prime numbers** no greater than n.
- Consider the following (imprecise) algorithm:
  - For each number i no greater than n, check whether it is a prime number.
- To check whether i is a prime number:
  - Idea: If any number $j < i$ can divide i, i is not a prime number.
  - Algorithm: For each number $j < i$, check **whether j divides i**. If there is any j that divides i, report no; otherwise, report yes.
- Before we write a program, we typically prefer to formalize our algorithm.
  - We write **pseudocodes**, a description of steps in words organized in a program structure.
  - This allows us to ignore the details of implementations.

Example: listing all prime numbers

- One pseudocode for listing all prime numbers no greater than *n* is:

```
Given an integer n:
for i from 2 to n
    assume that i is a prime number
    for j from 2 to i – 1
        if j divides i
            set i to be a composite number
    if i is still considered as prime
        print i
```

- Implementation:

```
for(int i = 2; i <= n; i++) {
  bool isPrime = true;
  for(int j = 2; j < i; j++) {
    if(i % j == 0) {
      isPrime = false;
      break;
    }
  }
  if(isPrime == true)
    cout << i << " ";
}
```

- Once we have described an algorithm in pseudocodes, implementation is easy.

A full implementation

- Let's **modularize** our implementation:
  - **`isPrime(int x)`** determines whether the given integer x is a prime number.

```
bool isPrime(int x)
{
  for(int i = 2; i < x; i++)
  {
    if(x % i == 0)
      return false;
  }
  return true;
}
```

- Now we have a correct algorithm.
  - May we improve this algorithm?

```
#include <iostream>
using namespace std;

bool isPrime(int x);
int main()
{
  int n = 0;
  cin >> n;

  for(int i = 2; i <= n; i++)
  {
    if(isPrime(i) == true)
      cout << i << " ";
  }
  return 0;
}
```

Improving our algorithm

- The algorithm can be **faster**:

```
bool isPrime(int x)
{
  for(int i = 2; i * i <= x; i++)
  {
    if(x % i == 0)
      return false;
  }
  return true;
}
```

  - Do not use **`i <= sqrt(x)`** (why?).
  - We improved the algorithm, **not** the implementation.
- May we do even better?

```
#include <iostream>
using namespace std;

bool isPrime(int x);
int main()
{
  int n = 0;
  cin >> n;

  for(int i = 2; i <= n; i++)
  {
    if(isPrime(i) == true)
      cout << i << " ";
  }
  return 0;
}
```

Improving our algorithm further

- Let's consider a completely different algorithm:
  - Let's start from 2. Actually 2, 4, 6, 8, … are all composite numbers.
  - For 3, actually 3, 6, 9, … are all composite numbers.
  - We may use a **bottom-up approach** to **eliminate composite numbers**.
- The pseudocode (with comments):

```
Given a Boolean array A of length n
Initialize all elements in A to be true // assuming prime
for i from 2 to n
    if A_i is true
        print i
        for j from 1 to ⌊n/i⌋  // eliminating composite numbers
            Set A[i × j] to false
```

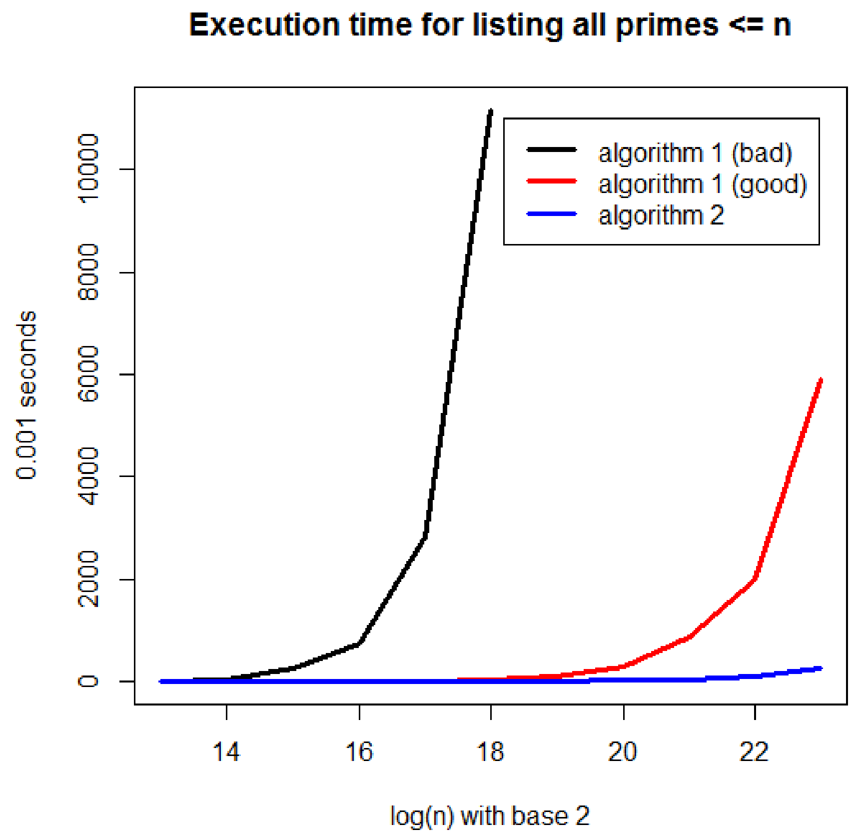
Improving our algorithm further

```
#include <iostream>
using namespace std;

const int MAX_LEN = 10000;

void ruleOutPrime
   (int x, bool isPrime[], int n);

int main()
{
  int n = 0;
  cin >> n; // must < 10000
  bool isPrime[MAX_LEN] = {0};
  for(int i = 0; i < n; i++)
    isPrime[i] = true;
```

```
  for(int i = 2; i <= n; i++)
  {
    if(isPrime[i] == true)
    {
      cout << i << " ";
      ruleOutPrime(i, isPrime, n);
    }
  }
  return 0;
}

void ruleOutPrime
  (int x, bool isPrime[], int n)
{
  for(int i = 1; x * i < n; i++)
    isPrime[x * i] = false;
}
```

Complexity

- While all the three algorithms are correct, they are not equally efficient.
- We typically care about the **complexity** of an algorithm:
  - **Time complexity**: the running time of an algorithm.
  - **Space complexity**: the amount of spaces used by an algorithm.
  - Time is typically more critical.
- Algorithm 2 is much faster!



Complexity

- Running time may be affected by the hardware, number of programs running at the same time, etc.
  - The **number of basic operations** is a better measurement.
  - Basic operations include simple arithmetic, comparisons, etc.
- Convince yourself that algorithm 2 does fewer basic operations.
- The calculation of complexity needs training.
  - This will be formally introduced in Discrete Mathematics, Data Structures, and/or Algorithms.

Outline

- Algorithms and complexity
- **Recursion**
- Searching and sorting

Recursive functions

- A function is **recursive** if it invokes itself (directly or indirectly).
- The process of using recursive functions is called **recursion**.
- Why recursion?
  - Many problems can be solved by dividing the original problem into one or several smaller pieces of **subproblems**.
  - Typically subproblems are **quite similar** to the original problem.
  - With recursion, we write one function to solve the problem by **using the same function** to solve subproblems.

Example 1: finding the maximum

- Suppose that we want to find the maximum number in an array $A[1..n]$ (which means A is of size n).
  - Is there any subproblem whose solution can be utilitzed?
  - Subproblem: Finding the maximum in an array with size smaller than n.
- A strategy:
  - Subtask 1: First find the maximum of $\boldsymbol{A[1..(n-1)]}$.
  - Subtask 2: Then compare that with $A[n]$.
- How would you visualize this strategy?
- While subtask 2 is simple, subtask 1 is **similar** to the original task.
  - It can be solved with the **same** strategy!

Example 1: finding the maximum

- Let's try to implement the strategy.
- First, I know I need to write a function whose header is:

```
double max(double array[], int len);
```

  - This function returns the maximum in **`array`** (containing **`len`** elements).
  - I **want** this to happen, though at this moment I do not know how.
- Now let's implement it:
  - If the function **really works**, subtask 1 can be completed by invoking

```
double subMax = max(array, len - 1);
```

  - Subtask 2 is done by comparing **`subMax`** and **`array[len - 1]`**.

Example 1: finding the maximum

- A (wrong) implementation:
- What will happen if we really invoke this function?
  - The program will not terminate!
  - Even when **`len`** is 1 in an invocation, we will still try to invoke **`max(array, 0)`**.
- For an array whose size is 1:
  - That number is the maximum!
- With this, we can add a **stopping condition** into our function.

```
double max(double array[], int len)
{
  double subMax = max(array, len - 1);
  if(array[len - 1] > subMax)
    return array[len - 1];
  else
    return subMax;
}

int main()
{
  double a[5] = {5, 7, 2, 4, 3};
  cout << max(a, 5);
  return 0;
}
```

Example 1: finding the maximum

- A correct implementation is:
- What is the outcome?

```
int main()
{
  double a[5] = {5, 7, 2, 4, 3};
  cout << max(a, 5);
  return 0;
}
```

- Both **`else`** can be removed. Why?

```
double max(double array[], int len)
{
  if(len == 1) // stopping condition
    return array[0];
  else
  {
    // recursive call
    double subMax = max (array, len - 1);
    if (array[len - 1] > subMax)
      return array[len - 1];
    else
      return subMax;
  }
}
```

Example 1: finding the maximum

- Is it okay to remove both **else**? Why?

```
double max(double array[], int len)
{
  if(len == 1) // stopping condition
    return array[0];
  else
  {
    // recursive call
    double subMax = max (array, len - 1);
    if(array[len - 1] > subMax)
      return array[len - 1];
    else
      return subMax;
  }
}
```

```
double max(double array[], int len)
{
  if(len == 1) // stopping condition
    return array[0];
  // recursive call
  double subMax = max (array, len - 1);
  if(array[len - 1] > subMax)
    return array[len - 1];
  return subMax;
}
```

Example 2: computing factorials

- How to write a function that computes the factorial of *n*?
  - A subproblem: computing the factorial of *n* – 1.
  - A strategy: First calculate the factorial of *n* – 1, then multiply it with *n*.

```
int factorial(int n)
{
  if(n == 1) // stopping condition
    return 1;
  else
    // recursive call
    return factorial(n - 1) * n;
}
```

Example 2: computing factorials

- When we invoke this function with argument 4:
- **`factorial(4)`**

```
= factorial(3) * 4
= (factorial(2) * 3) * 4
= ((factorial(1) * 2) * 3) * 4
= ((1 * 2) * 3) * 4
= (2 * 3) * 4
= 6 * 4
= 24
```

Example 3: the Fibonacci sequence

- Write a recursive function to find the *n*th Fibonacci number.
  - The Fibonacci sequence is 1, 1, 2, 3, 5, 8, 13, 21, …. Each number is the sum of the two proceeding numbers.
  - The *n*th value can be found once we know the (*n* – 1)th and (*n* – 2)th values.

```
int fib(int n)
{
  if(n == 1)
    return 1;
  else if(n == 2)
    return 1;
  else // two recursive calls
    return (fib(n - 1) + fib(n - 2));
}
```

Some remarks

- There must be a **stopping condition** in a recursive function. Otherwise, the program will not terminate.
- In many cases, a recursive strategy can also be implemented with **loops**.
  – E.g., writing a loop for finding a maximum and factorial.
  – But sometimes it is hard to use loops to imitate a recursive function.
- Compared with an equivalent iterative function, a recursive implementation is usually **simpler** and **easier to understand**.
- However, it generally uses **more memory spaces** and is **more time-consuming**.
  – Invoking functions has some cost.

Complexity issue of recursion

- In some cases, recursion is efficient enough.
  - E.g., finding a maximum or calculating the factorial.
- In some cases, however, recursion can be very **inefficient**!
  - E.g., Fibonacci.
- Let’s compare the efficiency of two different implementations.

Complexity issue of recursion

- Two implementations:

```
int fib(int n)
{
  if(n == 1)
    return 1;
  else if(n == 2)
    return 1;
  else // two recursive calls
    return (fib(n-1) + fib(n-2));
}
```

```
double fibRepetitive(int n)
{
  if(n == 1 || n == 2)
    return 1;
  int fib1 = 1, fib2 = 1;
  int fib3 = 0;
  for(int i = 2; i < n; i++)
  {
    fib3 = fib1 + fib2;
    fib1 = fib2;
    fib2 = fib3;
  }
  return fib3;
}
```

Complexity issue of recursion

- Which one is faster?

```
int main()
{
  int n = 0;
  cin >> n;
  cout << fibRepetitive(n) << "\n"; // algorithm 1
  cout << fib(n) << "\n"; // algorithm 2
  return 0;
}
```

Polynomial time vs. exponential time

- Given n:
  - The repetitive way has around $c_1 n$ steps, where $c_1 > 0$ is a constant.
  - The recursive way has around $c_2 2^n$ steps, where $c_2 > 0$ is a constant.
- When n is large enough, $c_2 2^n$ is much larger than $c_1 n$.
  - Even if $c_1 << c_2$!
  - We say the repetitive way is **more efficient**.
- Technically, we say that:
  - The repetitive way is a **polynomial-time** algorithm
  - The recursive way is an **exponential-time** algorithm.
- In general, an exponential-time algorithm is just too inefficient.

Power of recursion

- Though recursion is sometimes inefficient, typically implementation is easier.
- Let's consider the classic example "**Hanoi Tower**".
  - There are three pillars and disks of different sizes which can slide onto any pillar. Disc i is smaller than disc j if $i < j$.
  - A large disc cannot be placed on top of a small disc.
- Initially, all discs are at pillar A. We want to move them to pillar C:
  - Only one disk can be moved at a time.
  - Each move consists of taking the upper disk from one of the stacks and placing it on top of another stack.
- What are the steps that solve the Hanoi Tower problem in the fastest way?

A recursive implementation

```cpp
void hanoi(char from, char via,
           char to, int disc)
{
  if(disc == 1)
    cout << "From " << from
         << " to " << to << "\n";
  else
  {
    hanoi(from, to, via, disc - 1);
    cout << "From " << from
         << " to " << to << "\n";
    hanoi(via, from, to, disc - 1);
  }
}
```

```cpp
#include <iostream>
using namespace std;

int main()
{
  int disc = 0; // number of discs
  cin >> disc;
  char a = 'A', b = 'B', c = 'C';

  hanoi(a, b, c, disc);

  return 0;
}
```

- Is there a good way of solving the Hanoi Tower problem iteratively?

Outline

- Algorithms and complexity
- Recursion
- **Searching and sorting**

Searching

- One fundamental task in computation is to **search** for an element.
  - We want to determine whether an element exists in a set.
  - If yes, we want to locate that element.
  - E.g., looking for a string in an article.
- Here we will discuss how to search for an integer in an one-dimensional array.
- Whether the array is **sorted** makes a big difference.

Searching

- Consider an integer array $A[1..n]$ and an integer p.
- How to determine whether p exists in A?
- If so, where is it?
  - Assume that we only need to find one p even if there are multiple.
- Suppose that the array is unsorted.
- One of the most straightforward way is to apply a **linear search**.
  - Compare each element with p **one by one**, from the first to the last.
  - Whenever we find a match, report its location.
  - Conclude that p does not exist if we end up with nothing.
- The number of operations we need to execute is roughly proportional to n.

Binary search

- What if the array is sorted?
- We may still apply the linear search.
- However, we may improve the efficiency by implementing a **binary search**.
  - First, we compare p with the median m (e.g., $A[(n + 1) / 2]$ if n is odd).
  - If p equals m, bingo!
  - If $p < m$, we know p must exist in **the first half** of A if it exists.
  - If $p > m$, we know p must exist in **the second half** of A if it exists.
  - For the latter two cases, we will continue searching in the **subarray**.

Binary search: pseudocode

binarySearch(a sorted array A, search in between *from* and *to*, search for p)
if $n = 1$
  return true if $A_{from} = p$; ***return*** false otherwise
else
  let *median* be floor((*from* + *to*) / 2)
  if $p = A_{median}$
    return true
  else if $p < A_{median}$
    return binarySearch(A, ***from***, ***median***, p)
  else
    return binarySearch(A, ***median* + 1**, ***to***, p)

Linear search vs. binary search

- In binary search, the number of instructions to be executed is roughly proportional to $\log_2 n$.
- So binary search is **much more efficient** than linear search!
  - The difference is huge is the array is large.
  - However, binary search is possible only if the array is sorted.
  - Is it worthwhile to sort an array before we search it?
- It is natural to implement binary search with **recursion**.
  - A subproblem is to search for the element in one half of the array.
- Binary search can also be implemented with repetition.
  - Is it natural to do so?

Sorting

- Given a one-dimensional integer array A of size n, how to sort it?
- Given numbers 6, 9, 3, 4, and 7, how would you sort them?
- Recall what you typically do when you play poker:
  - First put the first number 6 aside.
  - Compare the second number 9 with 6. Because 9 > 6, put 9 to the right of 6.
  - Compare the third number 3 with the **sorted list** (6, 9). Because 3 < 6, put 3 to the left of 6.
  - Compare 4 with (3, 6, 9). Because 3 < 4 < 6, **insert** 4 in between 3 and 6.
  - Compare 7 with (3, 4, 6, 9). Because 6 < 7 < 9, insert 7 in between 6 and 9.
  - The result is (3, 4, 6, 7, 9).

Insertion sort

- The above algorithm is called **insertion sort**.
  - The key is to maintain a sorted list.
  - Then for each number in the unsorted list, **insert** it into the proper location so that the sorted list **remains sorted**.
- How would you implement the insertion sort?
  - Recursion or repetition?
  - If recursion, what is your strategy?

(Non-repetitive) insertion sort

- The pseudocode:

> insertionSort(a non-repetitive array A, the array length n, an index $cutoff < n$)
> // at any time, $A_{1..cutoff}$ is sorted and $A_{(cutoff + 1)..n}$ is unsorted
> ***if*** $A_{cutoff + 1} < A_{1..cutoff}$
>   let p be 1
> ***else***
>   find p such that $A_{p-1} < A_{cutoff + 1} < A_p$
> insert $A_{cutoff + 1}$ to A_p and shift $A_{p..cutoff}$ to $A_{(p + 1)..(cutoff + 1)}$
> ***if*** $cutoff + 1 < n$
>   insertionSort(A, n, $\boldsymbol{cutoff + 1}$)

- What if A is repetitive?

Insertion sort

- Roughly how many instructions do we need for insertion sort?
  - We need to do n insertions.
  - To insert the kth value, we search for a position and shift some elements.
    - A linear search: at most k comparisons.
    - Shifting: at most k shifts.
  - Roughly we need $1 + 2 + \cdots + n$ operations, which is proportional to n^2.
- Does binary search help?

Mergesort (Merge sort)

- Insertion sort is **simple** and fast!
  - Not really "fast", but faster than many similar sorting algorithm.
  - Because its idea and implementation is simple, it is faster than most algorithms when the array size is **small**.
- Interestingly, there is another sorting algorithm:
  - Its idea is somewhat similar to insertion sort.
  - But it is significantly faster for large arrays!
- This algorithm is called **mergesort**.

Mergesort (Merge sort)

- Recall that in an insertion sort, we need to insert one number into a sorted list for many times.
- A key observation is that "inserting" **another sorted list** of size *k* into a sorted list can be faster than inserting *k* separate numbers one by one!
  – So such "inserting" is actually "**merging**".
- Given an unsorted array, we will:
  – First split the array into two parts, the first half and second half.
  – Then sort each subarray.
  – Finally, merge these two subarrays.
- Mergesort is perfect for recursion!

Mergesort (Merge sort): pseudocode

mergeSort(an array A, the array length n)
  let *median* be floor((1 + n) / 2)
  mergeSort($A_{1..median}$, *median*) // now $A_{1..median}$ is sorted
  mergeSort($A_{(median+1)..n}$, $n - median + 1$) // now $A_{(median+1)..n}$ is sorted
  merge $A_{1..median}$ and $A_{(median+1)..n}$ // how?

Mergesort (Merge sort)

- Interestingly, insertion sort is a special way of running mergesort.
  - Not splitting the array into two halves.
  - Instead, splitting it into **A[1..n – 1]** and **A[n]**.
- Once we use the "smart split", the **efficiency** is improved a lot!
  - Insertion sort: Roughly proportional to n^2.
  - Merge sort: Roughly proportional to $n \log n$.
- A simple observation can make a huge difference!