GMBA 7098: Statistics and Data Analysis (Fall 2014)

Hypothesis testing (1)

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Introduction

- How do **scientists** (physicists, chemists, etc.) do research?
  - Observe phenomena.
  - Make hypotheses.
  - Test the hypotheses through experiments (or other methods).
  - Make conclusions about the hypotheses.
- In the business world, business researchers do the same thing with **hypothesis testing**.
  - One of the most important technique of statistical inference.
  - A technique for (statistically) **proving** things.
  - Again relies on **sampling distributions**.

Road map

- **Basic ideas of hypothesis testing**.
- The first example.
- The p-value.

People ask questions

- In the business (or social science) world, people ask questions:
  - Are older workers more loyal to a company?
  - Does the newly hired CEO enhance our profitability?
  - Is one candidate preferred by more than 50% voters?
  - Do teenagers eat fast food more often than adults?
  - Is the quality of our products stable enough?
- How should we answer these questions?
- Statisticians suggest:
  - First make a **hypothesis**.
  - Then **test** it with samples and statistical methods.

Hypotheses

- According to Merriam Webster's Collegiate Dictionary (tenth edition):
  - A **hypothesis** is a tentative explanation of a principle operating in nature.
- So we try to prove hypotheses to find reasons that explain phenomena and enhance decision making.

Statistical hypotheses

- A **statistical hypothesis** is a formal way of stating a hypothesis.
  - Typically with parameters and numbers.
- It contains two parts:
  - The **null hypothesis** (denoted as H_0).
  - The **alternative hypothesis** (denoted as H_a or H_1).
- The alternative hypothesis is:
  - The thing that we want (need) to prove.
  - The conclusion that can be made only if we have **a strong evidence**.
- The null hypothesis corresponds to a **default** position.

Statistical hypotheses: example 1

- In our factory, we produce packs of candy whose average weight should be 1 kg.
- One day, a consumer told us that his pack only weighs 900 g.
- We need to know whether this is just a rare event or our production system is out of control.
- If (we believe) the system is out of control, we need to shutdown the machine and spend two days for inspection and maintenance. This will cost us at least $100,000.
- So we should not to believe that our system is out of control just because of one complaint. What should we do?

Statistical hypotheses: example 1

- We may state a research hypothesis "Our production system is under control."
- Then we ask: Is there a strong enough evidence showing that the hypothesis is **wrong**, i.e., the system is out of control?
  - Initially, we assume our system is under control.
  - Then we do a survey for a "strong enough evidence".
  - We shutdown machines **only if** we prove that the system is out of control.
- Let μ be the average weight, the **statistical hypothesis** is

$$\mathrm{H}_0 : \mu = 1$$
$$\mathrm{H}_\mathrm{a} : \mu \neq 1.$$

Statistical hypotheses: example 2

- In our society, we adopt the presumption of innocence.
  - One is considered **innocent** until proven **guilty**.
- So when there is a person who probably stole some money:

$$\text{H}_0 : \text{The person is innocent}$$
$$\text{H}_a : \text{The person is guilty.}$$

- There are two possible errors:
  - One is guilty but we think she/he is innocent.
  - One is innocent but we think she/he is guilty.
- Which one is more critical?
  - It is unacceptable that an innocent person is considered guilty.
  - We will say one is guilty **only if** there is a strong evidence.

Statistical hypotheses: example 3

- Consider the research hypothesis "The candidate is preferred by more than 50% voters."
- As we need a default position, and the percentage that we care about is 50%, we will choose our null hypothesis as

$$H_0 : p = 0.5.$$

- How about the alternative hypothesis? Should it be

$$H_a : p > 0.5 \quad \text{or} \quad H_a : p < 0.5?$$

Statistical hypotheses: example 3

- The choice of the alternative hypothesis depends on the related **decisions** or **actions** to make.
- Suppose one will go for the election only if she thinks she will win (i.e., $p > 0.5$), the alternative hypothesis will be

$$\mathrm{H_a}: p > 0.5.$$

- Suppose one tends to participate in the election and will give up only if the chance is slim, the alternative hypothesis will be

$$\mathrm{H_a}: p < 0.5.$$

Remarks

- For setting up a statistical hypothesis:
  - Our default position will be put in the null hypothesis.
  - The thing we want to prove (i.e., the thing that needs a strong evidence) will be put in the alternative hypothesis.
- For writing the mathematical statement:
  - The **equal sign** ($=$) will always be put in the null hypothesis.
  - The alternative hypothesis contains an **unequal sign** or **strict inequality**: $\neq$, $>$, or $<$.
- The alternative hypothesis depends on the business context.

One-tailed tests and two-tailed tests

- If the alternative hypothesis contains an unequal sign ($\neq$), the test is a **two-tailed** test.
- If it contains a strict inequality ($>$ or $<$), the test is a **one-tailed** test.
- Suppose we want to test the value of the population mean.
  - In a two-tailed test, we test whether the population mean significantly deviates from a value. We do not care whether it is larger than or smaller than.
  - In a one-tailed test, we test whether the population mean significantly deviates from a value **in a specific direction**.

Road map

- Basic ideas of hypothesis testing.
- **The first example**.
- The p-value.

The first example

- Now we will demonstrate the process of hypothesis testing.
- Suppose we test the average weight (in g) of our products.

$$\begin{aligned} \mathrm{H}_0 &: \mu = 1000 \\ \mathrm{H}_\mathrm{a} &: \mu \neq 1000. \end{aligned}$$

- Once we have a strong evidence supporting H_a, we will claim that $\mu \neq 1000$.
- Suppose we know the variance of the weights of the products produced: $\sigma^2 = 40000 \text{ g}^2$.

Controlling the error probability

- Certainly the evidence comes from a **random** sample.
- It is natural that we may be **wrong** when we claim $\mu \neq 1000$.
  - E.g., it is possible that $\mu = 1000$ but we unluckily get a sample mean $\bar{x} = 912$.
- We want to **control the error probability**.
  - Let α be the maximum probability for us to make this error.
  - $1 - \alpha$ is called the **significance level**.
  - So if $\mu = 1000$, we will claim that $\mu \neq 1000$ with probability at most α.

Rejection rule

- Now let's test with the significance level $1 - \alpha = 0.95$.
- Intuitively, if $\overline{X}$ **deviates** from 1000 **a lot**, we should **reject** the null hypothesis and believe that $\mu \neq 1000$.
  - If $\mu = 1000$, it is so unlikely to observe such a large deviation.
  - So such a large deviation provides a **strong evidence**.
- So we start by sampling and calculating the **sample mean**.
  - Suppose the sample size $n = 100$.
  - Suppose the sample mean $\bar{x} = 963$.
- We want to construct a **rejection rule**: If $|\overline{X} - 1000| > d$, we reject H_0. We need to calculate d.

Rejection rule

$$H_0\colon \mu = 1000$$
$$H_a\colon \mu \neq 1000.$$

- We want a distance d such that

  if $\mathbf{H_0}$ is true, the probability of rejecting H_0 is 5%.

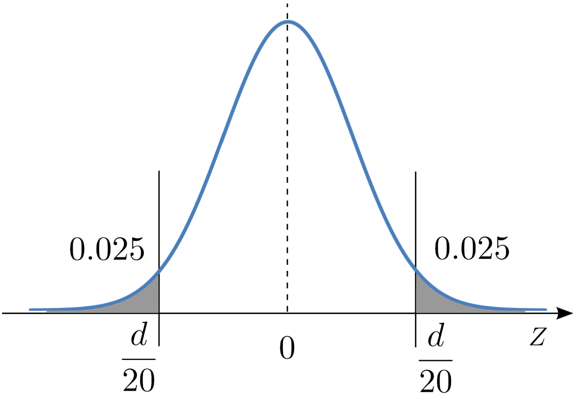
  - If H_0 is true, $\mu = 1000$. We reject H_0 if $|\overline{X} - 1000| > d$.
- Therefore, we need

$$\Pr\left(|\overline{X} - 1000| > d \middle| \mu = 1000\right) = 0.05.$$

  - People typically hide the condition $\mu = 1000$.
- The sample mean $\overline{X}$ has its sampling distribution.
  - Due to the central limit theorem, $\overline{X} \sim \mathrm{ND}(1000, 20)$.
  - This is under the assumption that $\mu = 1000$!

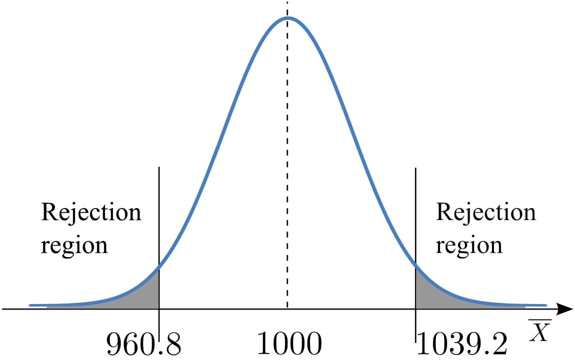
Rejection rule: the critical value

- $0.95 = \Pr(|\overline{X} - 1000| < d) = \Pr(1000 - d < \overline{X} < 1000 + d)$.



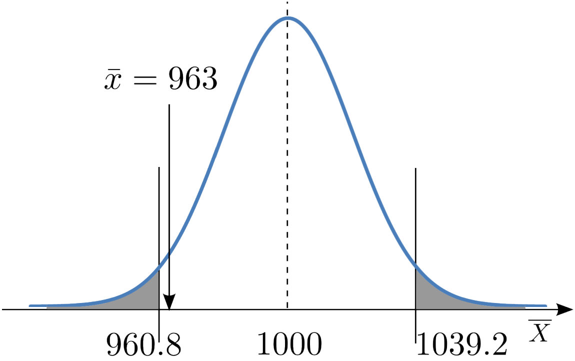
Rejection rule: the critical value

- The **rejection region** is $R = (-\infty, 960.8) \cup (1039.2, \infty)$.
- If $\overline{X}$ falls in the rejection region, we reject H_0.



Rejection rule: the critical value

- we **cannot reject H_0** because $\bar{x} = 963 \notin R$.
  - The deviation from 1000 is not large enough.
  - The evidence is not strong enough.



Rejection rule: the critical value

- In this example, the two values 960.8 and 1039.2 are the **critical values** for rejection.
  - If the sample mean is more extreme than one of the critical values, we reject H_0.
  - Otherwise, we do not reject H_0.
- $\bar{x} = 963$ is not strong enough to support H_a: $\mu \neq 1000$.
- Concluding statement:
  - Because the sample mean does not lie in the rejection region, we **cannot reject H_0**.
  - With a 95% significance level, there **is no** strong evidence showing that the average weight **is not** 1000 g.
  - Based on this result, we **should not** shutdown machines to do an inspection.

Summary

- We want to know whether H_0 is false, i.e., $\mu \neq 1000$.
- We control the probability of making a wrong conclusion.
  - If the machine is actually good, we do not want to reach a conclusion that requires an inspection and maintenance.
  - If H_0 ($\mu = 1000$) is true, we do not want to reject H_0.
  - We limit the probability at $\alpha = 5\%$.
- We will conclude that H_0 is false if the sample mean falls in the rejection region.
  - The calculation of the rejection region (i.e., the critical values) is based on **the z distribution**.
  - We conducted a **z test**.

Not rejecting vs. accepting

- We should be careful in writing our conclusions:
  - **Right**: Because the sample mean does not lie in the rejection region, we **cannot reject H_0**. With a 95% significance level, there **is no** strong evidence showing that the average weight **is not** 1000 g.
  - **Wrong**: Because the sample mean does not lie in the rejection region, we **accept** H_0. With a 95% significance level, there **is** a strong evidence showing that the average weight **is** 1000 g.
  - Unable to prove one thing is false does not mean it is true!

The first example (part 2)

- Suppose that we modify the hypothesis into a directional one:[1]

$$H_0 : \mu = 1000.$$
$$H_a : \mu < 1000.$$

We still have $\sigma^2 = 40000$, $n = 100$, $\alpha = 0.05$.

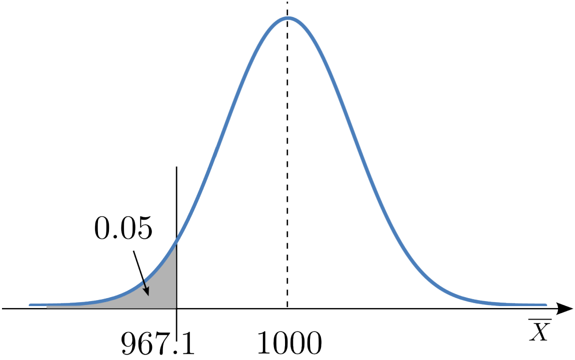
  - This is a one-tailed test.
  - Once we have a strong evidence supporting $\mathrm{H_a}$, we will claim that $\mu < 1000$.
  - We need to find a distance d such that

$$\Pr\left(1000 - \overline{X} > d \middle| \mu = 1000\right) = 0.05.$$

[1]Some researchers write $\mu \geq 1000$ in this case.

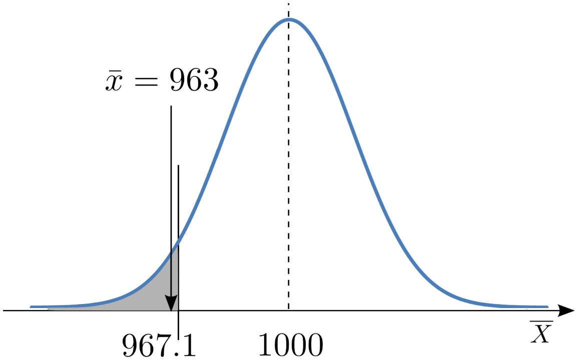
Rejection rule: the critical value

- We need $0.05 = \Pr(1000 - \overline{X} > d)$.
  - The critical value $d = 1.645 \times 20 = 32.9$.
  - The rejection region is $(-\infty, 967.1)$.



Rejection rule: the critical value

- As the observed sample mean $\bar{x} = 963 \in (-\infty, 967.1)$, we **reject H_0**.
  - The deviation from 1000 is large enough.
  - The evidence is strong enough.



Rejection rule: the critical value

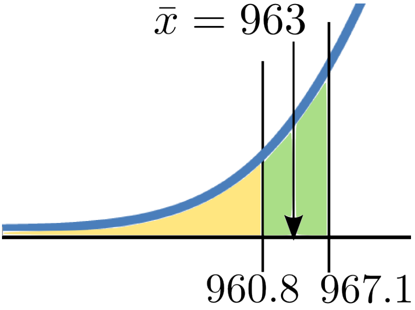
- In this example, 967.1 is the critical values for rejection.
  - If the sample mean is more extreme than (in this case, below) the critical value, we reject H_0.
  - Otherwise, we do not reject H_0.
- There is a strong evidence supporting H_a: $\mu < 1000$.
- Concluding statement:
  - Because the sample mean lies in the rejection region, we **reject H_0**. With a 95% significance level, there **is a** strong evidence showing that the average weight **is less than** 1000 g.

One-tailed tests vs. two-tailed tests

- When should we use a two-tailed test?
  - We should use a two-tailed test to be **conservative**.
  - E.g., we suspect that the parameter **has changed**, but we are **unsure** whether it becomes larger or smaller.
- If we know or believe that the change is possible **only in one direction**, we may use a one-tailed test.
- If we do not know it, using one-tailed test is **dangerous**.
  - In the previous example with $H_a : \mu < 1000$.
  - If $\bar{x} = 2000$, all we can say is "there is no strong evidence that $\mu < 1000$."
  - We are unable to conclude that $\mu \neq 1000$.

One-tailed tests vs. two-tailed tests

- Having more information (i.e., knowing the direction of change) makes rejection "easier".
- Easier to find a strong enough evidence.



Summary

- Distinguish the following pairs:
  - One- and two-tailed tests.
  - No evidence showing H_0 is false and having evidence showing H_0 is true.
  - Not rejecting H_0 and accepting H_0.
  - Using $=$ and using $\geq$ or $\leq$ in the null hypothesis.

Road map

- Basic ideas of hypothesis testing.
- The first example.
- **The p-value.**

The p-value

- The **p-value** is an important, meaningful, and widely-adopted tool for hypothesis testing.

Definition 1

In a hypothesis testing, for an observed value of the statistic, the p-value is the probability of observing a value that is at least as extreme as the observed value under the assumption the null hypothesis is true.

  - Based on an **observed** value of the statistic.
  - Is the **tail probability** of the observed value.
  - Assuming that the null hypothesis is true.

The p-value

- Mathematically:
  - Suppose we test a population mean μ with a one-tailed test

$$H_0 : \mu = 1000$$
$$H_a : \mu < 1000.$$

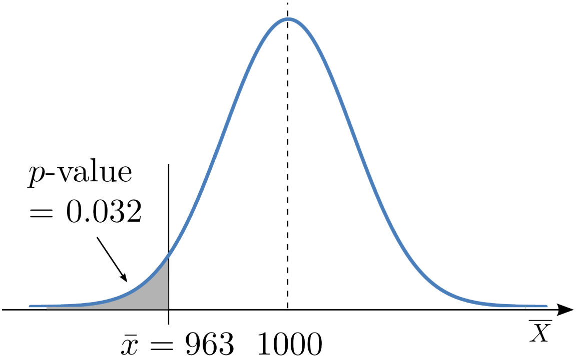
  - Given an observed $\bar{x}$, the p-value is defined as

$$\Pr(\overline{X} < \bar{x}).$$

- In the previous example:
  - $\sigma^2 = 40000$, $n = 100$, $\alpha = 0.05$, $\bar{x} = 963$.
  - How to calculate the p-value of $\bar{x}$?

The p-value

- If H_0 is true, i.e., $\mu = 1000$, we have:
  - $\Pr(\overline{X} \leq 963) = 0.032$.



How to use the p-value?

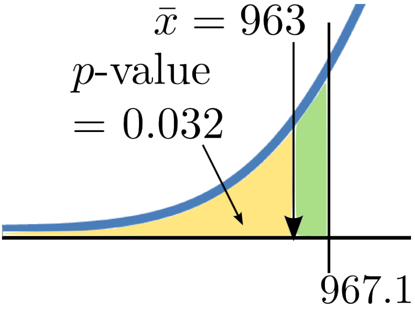
- The p-value can be used for constructing a **rejection rule**.
- For a one-tailed test:
  - If the p-value is **smaller** than α, we **reject** H_0.
  - If the p-value is greater than α, we do not reject H_0.
- Consider the one-tailed test

$$H_0: \mu = 1000$$
$$H_a: \mu < 1000.$$

  - Suppose we still adopt $\alpha = 0.05$.
  - Because the p-value $0.032 < 0.05$, we reject H_0.

p-values vs. critical values

- Using the p-value is **equivalent** to using the critical values.
  - The rejection-or-not decision we make will be the same based on the two methods.



The benefit of using the p-value

- In calculating the p-value, we do not need α.
- After the p-value is calculated, we compare it with α.
- The p-value, which needs to be calculated **only once**, allows us to know whether the evidence is strong enough under various significance levels.

α	0.1	0.05	0.01
Rejecting H_0?	Yes ($0.032 < 0.1$)	Yes ($0.032 < 0.05$)	No ($0.032 > 0.01$)

  - If we use the critical-value method, we need to calculate the critical value for three times, one for each value of α.

The benefit of using the p-value

- In many studies, the researchers do not determine the significance level $1 - \alpha$ before a test is conducted.
- They calculate the p-value and then mark **how significant** the result is with **stars**.

p-value	< 0.01	< 0.05	< 0.1	> 0.1
Significant?	Highly significant	Moderately significant	Slightly significant	Insignificant
Mark	***	**	*	(Empty)

The benefit of using the p-value

- As an example, suppose one is testing whether people sleep at least eight hours per day in average.
  - Age groups: $[10, 15)$, $[15, 20)$, $[20, 35)$, etc.
  - For group i, a one-tailed test is conducted. $\text{H}_\text{a}: \mu_i > 8$.
  - The result may be presented in a table:

Group	Age group	p-value
1	[10,15)	0.002***
2	[15,20)	0.2
3	[20,25)	0.06*
4	[25,30)	0.04**
5	[30,35)	0.03**

- A smaller p-value does NOT mean **a larger deviation**!
  - We cannot conclude that $\mu_5 > \mu_4$, $\mu_1 > \mu_3$, etc.

The p-value for two-tailed tests

- How to construct the rejection rule for a **two-tailed** test?
  - If the p-value is **smaller** than $\frac{\alpha}{2}$, we **reject** H_0.
  - If the p-value is greater than $\frac{\alpha}{2}$, we do not reject H_0.
- Consider the two-tailed test

$$H_0 : \mu = 1000.$$
$$H_a : \mu \neq 1000.$$

  - Suppose we still adopt $\alpha = 0.05$.
  - Because the p-value $0.032 > \frac{\alpha}{2} = 0.025$, we do not reject H_0.

Summary

- The p-value is the tail probability of the realization of a statistics assuming the null hypothesis is true.
- The p-value method is an alternative way of making the rejection decision.
  - It is equivalent to the critical-value method.
- The p-value is related to how likely for H_0 to be false.
- It does not measure how larger the deviation is.