Statistics and Data Analysis

Introduction to Probability (1)

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An example of statistical inference

- Quality control: For all LED lamps of brand IM, we are interested in μ, the average number of hours of luminance.
- Let's select a random sample of 40 lamps. A test shows that the sample average is $\bar{x} = 28000$ hours.
  - If I estimate that $\mu = 28000$, how likely I will be right?
  - If I estimate that $\mu \in [27000, 29000]$, how likely I will be right?
  - How about $\mu \in [26000, 30000]$?
- To assess these probabilities, we need to study Probability.

Road map

- **Basic concepts**.
- Independent events.
- Random variables.
- Descriptive measurements.

Experiments and events

- An **experiment** is a process that produces **(random) outcomes**.
  - Tossing a coin. Outcomes: head or tail.
  - Testing a new drug on a patient: Outcomes: Effective, not effective, getting worse.
  - Interviewing 20 consumers regarding how many will buy a new product. Outcomes: 10, 15, 0, etc.
  - Sampling every 200th bottle of ketchup for its weight. Outcome?
- An **event** is an outcome of an experiments.
- Each event has its **probability** to occur.
  - Tossing a fair coin: $\frac{1}{2}$ for head and $\frac{1}{2}$ for tail.
  - Rolling a fair dice: $\frac{1}{6}$ for each possible outcome.
- Let A be an event of an experiment, we write $\Pr(A)$ to denote the probability for A to occur.
  - Let A be getting a head when tossing a fair coin, then $\Pr(A) = \frac{1}{2}$.

Elementary events

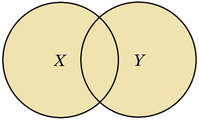
- An **elementary event** is an event that **cannot be decomposed** into smaller events.
- Consider the experiment of rolling a dice.
  - Getting 3 is an elementary event.
  - How about getting a number larger than 3?
  - The event of getting larger than 3 can be decomposed into three elementary events: getting 4, 5, and 6.
  - How about getting an even number?
- For asking Jane, Mary, Melissa, and Lucy about a new product:
  - Is "one is willing to buy" an elementary event?
  - How about "Mary is willing to buy but all the other three are not?"

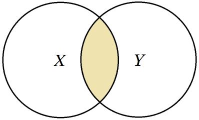
Sample spaces

- The **sample space** of an experiment is the collection of all elementary events.
  - A sample space contains "all basic things that may happen."
  - Nothing outside the sample space can occur.
- What is the sample space of:
  - Rolling a dice?
  - Rolling two dices?
  - Asking 20 consumers?
  - Testing a new drug?
- If S is a sample space, we have $\Pr(S) = 1$.
- A sample space is a **set**. Elementary elements are **elements** of the set. Events are **subsets** of the set.
  - If x is an elementary event of an event X, we write $x \in X$.
  - E.g., "getting 2" $\in$ "getting an even number."

Unions and intersections

- Let A and B be two events and S be the sample space.
- The **union** of A and B, denoted by $A \cup B$, contains elementary events in A **or** B.
  - $A \cup B = \{x | x \in A \text{ or } x \in B\}$.
  - E.g., $\{2, 3, 5\} \cup \{1, 5, 6\} = \{1, 2, 3, 5, 6\}$.
- The **intersection** of A and B, denoted by $A \cap B$, contains elementary events that are in A **and** B.
  - $A \cap B = \{x | x \in A \text{ and } x \in B\}$.
  - E.g., $\{2, 3, 5\} \cap \{1, 5, 6\} = \{5\}$.





Unions and intersections

- The union of two (or more) events is also an event.
  - Consider rolling a fair dice.
  - Let event A be getting an even number. We have $\Pr(A) = \frac{1}{2}$.
  - Let event B be getting larger than three. We have $\Pr(B) = \frac{1}{2}$.
  - The **union probability** of A and B is

$$\Pr(A \cup B) = \Pr(\text{getting } 2, 4, 5, \text{ or } 6) = \frac{2}{3}.$$

- The intersection of two (or more) events is also an event.
  - Consider rolling a fair dice.
  - The **joint probability** of A and B is

$$\Pr(A \cap B) = \Pr(\text{getting } 4 \text{ or } 6) = \frac{1}{3}.$$

- In fact, A and B are both unions of multiple elementary events.

Two special cases

- Events are **mutually exclusive** if there is no intersection.
  - $A \cap B = \emptyset$ (empty).
  - Events are mutually exclusive if all their elementary events are different.
  - E.g., for rolling a dice, getting an even number and getting 5 are mutually exclusive.
- Events are **collectively exhaustive** if they together cover the whole sample space.
  - $S = A \cup B$.
  - Events are collectively exhaustive if one of them must occur.
  - E.g., for rolling a dice, getting an even number and getting smaller than six are collectively exhaustive.
  - Two collectively exhaustive sets are **not necessarily** mutually exclusive!

Complements

- The **complement** of X, denoted by X', contains all elements not contained in X.
  - $X' = \{x | x \notin X\}$, where $x \notin X$ means x is not an element of X.
  - Graphically:
  - E.g., for rolling a dice, getting less than three and getting greater than two are complements.
  - E.g., for rolling a dice, getting less than three and getting greater than three are not complements.
- For any set X, X and its complement X' are mutually exclusive and collectively exhaustive, i.e., $X \cap X' = \emptyset$ and $X \cup X' = S$.
- Intuitively, $\Pr(X') = 1 - \Pr(X)$.

Road map

- Basic concepts.
- **Independent events**.
- Random variables.
- Descriptive measurements.

Independent events

- Two events are **independent** if **whether one occurs** does not affect **the probability** for the other one to occur.
- Two events are **dependent** if they are not independent.
- A set of events are independent if **all** pairs of events are independent.
- Are the following pairs of events independent?
  - Rolling two today and rolling three tomorrow with a fair dice.
  - A customer is a man and he likes watching baseball.
  - One's phone number contains "7" and she was born on July.
  - A laptop is defective and it has a 14-inch screen.

Mathematical property

- For independent events, calculating the joint probability is easy:

Proposition 1

For any two independent events A and B, we have

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

- E.g., suppose we toss an unfair coin whose probability of head is $\frac{2}{3}$.
  - Let H be getting a head and T be getting a tail in one toss: $\Pr(H) = \frac{2}{3}$ and $\Pr(T) = \frac{1}{3}$.
  - Let HH be getting two heads, TT be getting two tails, HT be getting a head then a tail, and TH be getting a tail then a head in two tosses:

$$\Pr(HH) = \Pr(H)\Pr(H) = \frac{4}{9}, \Pr(HT) = \Pr(H)\Pr(T) = \frac{2}{9}, \text{etc.}$$

Joint probability tables

- Two experiments may be presented by a **joint probability table**.
  - Events of experiment 1 are listed in the first **column**.
  - Events of experiment 2 are listed in the first **row**.
  - A column and a row at the margin for **totals**.
- For the previous example of an unfair dice:

1st	2nd		Total
	H	T	
H	?	?	$\frac{2}{3}$
T	?	?	$\frac{1}{3}$
Total	$\frac{2}{3}$	$\frac{1}{3}$	1

  - The last column records the probabilities of H and T for the first toss.
  - The last row records the probabilities of H and T for the second toss.
- How to find the joint probabilities?

Calculating joint probabilities

- To find the **joint probabilities** of two independent events A and B, simply apply $\Pr(A \cap B) = \Pr(A)\Pr(B)$.
  - For the previous example of an unfair dice:

1st	2nd H	2nd T	Total
H	$\frac{4}{9}$	$\frac{2}{9}$	$\frac{2}{3}$
T	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
Total	$\frac{2}{3}$	$\frac{1}{3}$	1

  - Each entry records a joint probability.
- Two joint events corresponding to two entries are mutually exclusive.
  - The **union probability** can be found by summing up joint probabilities.
  - E.g., the probability of "getting exactly one head" is

$$\Pr(HT \text{ or } TH) = \frac{2}{9} + \frac{2}{9} = \frac{4}{9}.$$

Joint probability tables with dependent events

- Events are not always independent.

	Supporting KMT	Supporting DPP	Neither
Will vote for Ko	17%	85%	37%
Will vote for Lien	71%	4%	20%

	Women	Men
Will vote for Ko	36%	39%
Will vote for Lien	54%	30%

(http://www.chinatimes.com/newspapers/20140929000800-260302)

Road map

- Basic concepts.
- Independent events.
- Random variables.
- **Descriptive measurements**.

Random variables

- A **random variable** (RV) is a variable whose outcomes are random.
- Examples:
  - The outcome of tossing a coin.
  - The outcome of rolling a dice.
  - The number of people preferring Pepsi to Coke in a group of 25 people.
  - The number of consumers entering a store at 7-8pm.
  - The temperature of this classroom at tomorrow noon.
  - The average studying hours of a group of 10 students.

Discrete and continuous random variables

- A random variable can be discrete or continuous.
- For a discrete RV, its value is **counted**.
  - The outcome of tossing a coin.
  - The outcome of rolling a dice.
  - The number of people preferring Pepsi to Coke in a group of 25 people.
  - The number of consumers entering a store at 7-8pm.
- For a continuous RV, its value is **measured**.
  - The temperature of this classroom at tomorrow noon.
  - The average studying hours of a group of 10 students.
- A discrete RV has **gaps** among its possible values; a continuous RV's possible values typically form an **interval**.

Discrete and continuous distributions

- How to describe a random variable?
  - Writing down all possible values (the **sample space**) is not enough.
  - For each possible value, we must also describe **how likely** it will occur.
- The likelihoods for all outcomes of a random variable to be realized are summarized by **probability distributions**, or simply distributions.
- As variables can be either discrete or continuous, distributions may also be either discrete or continuous.
  - Today we study discrete distributions.
  - In the next week we study continuous distributions.

Describing a discrete distribution

- For a discrete random variable, we may **list** all possible outcomes and their probabilities.
  - Let X be the result of tossing a fair coin:

x	Head	Tail
$\Pr(X = x)$	$\frac{1}{2}$	$\frac{1}{2}$

  - Let X be the result of rolling a fair dice:

x	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

- The function $\Pr(X = x)$, sometimes abbreviated as $\Pr(x)$, for all $x \in S$, is called the **probability mass function** (pmf) or probability function of X.
- For any random variable X, we have $\sum_{x \in S} \Pr(X = x) = 1$.

Describing a discrete distribution: an example

- Let X_1 be the result of tossing a fair coin for the first time.
- Let X_2 be the result of tossing a fair coin for the second time.
- Let Y be the **number of heads** obtained by tossing a fair coin twice.
- What is the distribution of Y?
  - Possible values: 0, 1, and 2.
  - Probabilities: What are $\Pr(Y = 0)$, $\Pr(Y = 1)$, and $\Pr(Y = 2)$?
- According to the joint probability table:

	$X_2 = \text{Head}$	$X_2 = \text{Tail}$
$X_1 = \text{Head}$	$\frac{1}{4}$	$\frac{1}{4}$
$X_1 = \text{Tail}$	$\frac{1}{4}$	$\frac{1}{4}$

y	0	1	2
$\Pr(Y = y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

- How would you find the distribution of Z, the number of heads obtained by tossing a fair coin for three times?

Road map

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- **Descriptive measurements**.

Descriptive measurements

- Consider a discrete random variable X with a sample space S, realizations $\{x_i\}_{i \in S}$, and a pmf $\Pr(\cdot)$.
- The **expected value** (or mean) of X is

$$\mu \equiv \mathbb{E}[X] = \sum_{i \in S} x_i \Pr(x_i).$$

- The **variance** of X is

$$\sigma^2 \equiv \text{Var}(X) \equiv \mathbb{E}\big[(X - \mu)^2\big] = \sum_{i \in S} (x_i - \mu)^2 \Pr(x_i).$$

- The **standard deviation** of X is $\sigma \equiv \sqrt{\sigma^2}$.

Descriptive measurements: example 1

- Let X be the outcome of rolling a dice, then the pmf is $\Pr(x) = \frac{1}{6}$ for all $x = 1, 2, ..., 6$.
  - The expected value of X is
$$\mathbb{E}[X] \equiv \sum_{i=1}^{6} x_i \Pr(x_i) = \frac{1}{6}(1 + 2 + \cdots + 6) = 3.5.$$
  - The variance of X is
$$\begin{aligned} \mathrm{Var}(X) &\equiv \sum_{i \in S} (x_i - \mu)^2 \Pr(x_i) \\ &= \frac{1}{6}\left[(-2.5)^2 + (-1.5)^2 + \cdots + 2.5^2\right] \approx 2.92. \end{aligned}$$
  - The standard deviation of X is $\sqrt{2.92} \approx 1.71$.

Descriptive measurements: example 2

- Let X be the outcome of rolling an unfair dice:

x_i	1	2	3	4	5	6
$\Pr(x_i)$	0.2	0.2	0.2	0.15	0.15	0.1

  - The expected value of X is

$$\begin{aligned}\mathbb{E}[X] &\equiv \sum_{i=1}^{6} x_i \Pr(x_i) \\ &= 1 \times 0.2 + 2 \times 0.2 + 3 \times 0.2 + 4 \times 0.15 + 5 \times 0.15 + 6 \times 0.1 \\ &= 3.15.\end{aligned}$$

  - Note that $3.15 < 3.5$, the expected value of rolling a fair dice. Why?

Descriptive measurements: example 2

- Let X be the outcome of rolling an unfair dice:

x_i	1	2	3	4	5	6
$\Pr(x_i)$	0.2	0.2	0.2	0.15	0.15	0.1

  - The expected value of X is $\mu = 3.15$.
  - The variance of X is

$$\begin{aligned} \text{Var}(X) &\equiv \sum_{i \in S} (x_i - \mu)^2 \Pr(x_i) \\ &= (-2.15)^2 \times 0.2 + (-1.15)^2 \times 0.2 + (-0.15)^2 \times 0.2 \\ &\quad + 0.85^2 \times 0.15 + 1.85^2 \times 0.15 + 2.85^2 \times 0.1 \\ &\approx 2.6275. \end{aligned}$$

  - Note that $2.6275 < 2.92$, the variance of rolling a fair dice. Why?
  - The standard deviation of X is $\sqrt{2.6275} \approx 1.62$.