Statistics and Data Analysis
Hypothesis Testing

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1. For each of the following scenario, write down the statistical hypothesis. In particular, state the null and alternative hypothesis.

   (a) Food and Drug Administration (FDA) examines the effectiveness of a new drug. Let p_1 and p_2 be the survival rates of a patient taking and not taking this drug. The drug will be approved only if $p_1 > p_2$.

   (b) An instructor determines whether two students copy each other's homework.

   (c) A company decides whether to enter a new market.
   Hint. What else do we need to know to state the hypothesis?

2. We are interested in knowing whether a coin is fair. Let p be the probability of obtaining a head and $q = 1 - p$ be that of obtaining a tail. Let X_i be the outcome of the ith toss, where $X_i = 1$ means a head and $X_i = 0$ means a tail. Let $S_n = \sum_{i=1}^{n} X_i$ be the number of heads observed in n tosses.

(a) Find $\Pr(S_2 = j)$ for $j \in \{0, 1, 2\}$.

(b) Convince yourself that

$$\Pr(S_3 = 0) = q^3,$$
$$\Pr(S_3 = 1) = pqq + qpq + qqp = 3pq^2,$$
$$\Pr(S_3 = 2) = ppq + pqp + ppq = 3p^2q, \text{ and}$$
$$\Pr(S_3 = 3) = p^3.$$

(c) Find $\Pr(S_4 = j)$ for $j \in \{0, 1, 2, 3, 4\}$.

(d) Convince yourself that

$$\Pr(S_n = j) = \binom{n}{j} p^j (1-p)^{n-j} \quad \forall j \in \{0, 1, ..., n\},$$

where

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

is the number of of combinations of choosing j out of n items.

Note. The MS Excel function `COMBIN(n, j)` returns $\binom{n}{j}$.

(e) Let $n = 20$ and $p = 0.4$, find $\Pr(S_n = j)$ for all possible values of j. Depict these probabilities to visualize the distribution of S_n.

Note. S_n is said to follow the *binomial distribution* with n trials and success probability p.

3. We are interested in knowing whether a coin is fair. Let p be the probability of obtaining a head. Let S_n be the number of heads observed in n tosses. Note that S_n is a statistics (sample sum).

(a) Our default position is that "the coin is fair." This gives us a null hypothesis

$$H_0 : p = 0.5.$$

What is the alternative hypothesis if we want to test whether the coin is fair?

(b) Suppose that we observe 7 heads in 10 tosses. Under our null hypothesis, find the probability $\Pr(S_{10} \geq 7)$.

(c) Consider the definition of p-value on page 30 of the slides, convince yourself that $\Pr(S_{10} \geq 7)$ is the p-value of observing 7 heads in 10 tosses.

(d) Suppose that we observe 7 heads in 10 tosses. What is our conclusion about our hypothesis with a 95% confidence level?

(e) What if we observe 9 heads in 10 tosses?

(f) What if we observe 70 heads in 100 tosses?

(g) The idea of using a small p-value is to reject a null hypothesis is "*if the null hypothesis is true, the probability of observing such an extreme value is at most the p-value. As the p-value is so small, it is quite unlikely that the null hypothesis is true.*" Review this idea and apply them to the above problems.

(h) For a sample of 100 tosses and a 95% confidence level, find a value d such that we reject the null hypothesis when $S_n \geq 50 + d$ or $S_n \leq 50 - d$.

Note. This is the concept of the rejection region.

4. The national average price charged to clean a 12' by 18' carpet is $50. A start-up carpet-cleaning company believes that in their target region, the average price for this service is higher. If this is true, it will start its business in this region. 25 random customers who have recently had a 12' by 18' carpet cleaned were asked about the prices they paid. See the sheet "Carpet" for the data. Suppose the population standard deviation is known to be $3.25, and these prices are normally distributed in the population. Let $\alpha = 0.05$.

(a) Write down a statistical hypothesis for testing the average price in this region.

(b) Find the critical value(s) for the rejection region.

(c) Find the p-value of the test.

(d) What should be their conclusion? What if $\alpha = 0.01$?