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Suggested Solution for Homework 04

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1. (a) First we construct the joint probability table in Table 1, where WE, WL, WVL, OT, and L means waking up early, waking up late, waking up very late, being on time, and being late. Then the probability that Bob will be late is 0.433.

	OT	L	Total
WE	0.300	0.033	0.333
WL	0.200	0.133	0.333
WVL	0.067	0.267	0.333
Total	0.567	0.433	1.000

Table 1: Calculations for Problem 1a.

	OT	L	Total
WE	0.069	0.008	0.077
WL	0.185	0.123	0.308
WVL	0.123	0.492	0.615
Total	0.377	0.623	1.000

Table 2: Calculations for Problem 1c.

(b) The posterior belief on waking up early is $\Pr(WE|L) = \frac{\Pr(WE \cap L)}{\Pr(L)} \approx \frac{0.033}{0.433} \approx 0.077$, on waking up late is $\Pr(WL|L) = \frac{\Pr(WL \cap L)}{\Pr(L)} \approx \frac{0.133}{0.433} \approx 0.308$, and on waking up very late is $\Pr(WVL|L) = \frac{\Pr(WVL \cap L)}{\Pr(L)} \approx \frac{0.133}{0.433} \approx 0.615$,

(c) **Method 1.** We first focus on the probability of being late twice. By assuming that what happens on two days are independent, we have $\Pr(L \cup L|WE) = \Pr(L|WE)\Pr(L|WE) = 0.01$. Similarly, we have $\Pr(L \cup L|WL) = 0.16$ and $\Pr(L \cup L|WVL) = 0.64$. It then follows that the posterior belief on waking up early is

$$\Pr(WE|L \cup L) = \frac{\Pr(WE)\Pr(L \cup L|WE)}{\displaystyle\sum_{e \in \{WE, WL, WVL\}} \Pr(e)\Pr(L \cup L|e)} = \frac{0.01}{0.81} \approx 0.012.$$

Similarly, we have $\Pr(WL|L \cup L) = \frac{0.016}{0.81} \approx 0.198$, and $\Pr(WVL|L \cup L) = \frac{0.64}{0.81} \approx 0.79$.
Method 2. The second way is to apply the updating in Part (b) again. To do so, we should use the posterior believes after the first day as the prior belief before the second day, i.e., put 0.077, 0.308, and 0.615 at the "Total" column. We may then complete the new joint probability table, as shown in Table 2, and then find the posterior believes after the second day. The posterior believes are $\frac{0.008}{0.623} \approx 0.012$ for waking up early, $\frac{0.123}{0.623} \approx 0.198$ for waking up late, and $\frac{0.492}{0.623} \approx 0.79$ for waking up very late.

(d) While we may still apply the two methods described in Part (c), here only the second method is used. By using 0.012, 0.198, and 0.79 as the prior believes, we may compete the third joint probability table, as shown in Table 3. It then follows that the posterior believes are $\frac{0.011}{0.288} \approx 0.039$ for waking up early, $\frac{0.119}{0.288} \approx 0.412$ for waking up late, and $\frac{0.158}{0.288} \approx 0.549$ for waking up very late.

	OT	L	Total
WE	0.011	0.001	0.012
WL	0.119	0.079	0.198
WVL	0.158	0.632	0.790
Total	0.288	0.712	1.000

Table 3: Calculations for Problem 1d.

xy	$\Pr(xy)$	xy	$\Pr(xy)$
0	0.235	6	0.120
1	0.100	8	0.035
2	0.190	9	0.050
3	0.150	12	0.025
4	0.095	–	–

Table 4: Pmf of XY for Problem 2f.

2. (a) $\mathbb{E}[Y] = \sum_{y=0}^{3} y \Pr(Y = y) = 1.7$ (in MS Excel, you may use the function SUMPRODUCT()).

(b) $\text{Var}(Y) = \sum_{y=0}^{3}(y - \mathbb{E}[X])^2 \Pr(Y = y) = 1.01$.

(c) $\mathbb{E}[2 - 4Y] = 2 - 4\mathbb{E}[Y] = -4.8$.

(d) $\text{Var}(2 - 4Y) = 16\text{Var}(Y) = 16.16$.

(e) No formula can be used for solving the mean of Y^2 directly, so we solve it by definition. Note that Y^2 is also a random variable with its own pmf. Therefore,

$$\mathbb{E}[Y^2] = \sum_{y=0}^{3} y^2 \Pr(Y = y) = 0 \times 0.15 + 1 \times 0.25 + 4 \times 0.35 + 9 \times 0.25 = 3.9.$$

3. (a) Because X and Y are independent, we have

$$\Pr(X + Y = 3) = \sum_{x=0}^{3} \Pr(X = x, Y = 3 - x) = \sum_{x=0}^{3} \Pr(X = x)\Pr(Y = 3 - x)$$
$$= 0.1 \times 0.25 + 0.4 \times 0.35 + 0.2 \times 0.25 + 0.2 \times 0.15 = 0.245.$$

In general, calculating $\Pr(X + Y = k)$ for any k requires a complete enumeration like this.

(b) Again, because X and Y are independent, we have

$$\Pr(XY = 4) = \sum_{x \in \{2,4\}} \Pr\left(X = x, Y = \frac{4}{x}\right) = \sum_{x \in \{2,4\}} \Pr(X = x)\Pr\left(Y = \frac{4}{x}\right)$$
$$= 0.2 \times 0.35 + 0.1 \times 0.25 = 0.095.$$

In general, calculating $\Pr(XY = k)$ for any k requires a complete enumeration like this.

(c) $\mathbb{E}[3X + Y] = 3\mathbb{E}[X] + \mathbb{E}[Y] = 3 \times 1.8 + 1.7 = 7.1$, where $\mathbb{E}[X] = \sum_{x=0}^{4} x \Pr(X = x) = 1.8$.

(d) Because X and Y are independent, we have

$$\mathbb{E}[6 + XY] = 6 + \mathbb{E}[XY] = 6 + \mathbb{E}[X]\mathbb{E}[Y] = 6 + 1.8 \times 1.7 = 9.06.$$

(e) Because X and Y are independent, we have

$$\text{Var}(X - 2Y) = \text{Var}(X) + 4\text{Var}(Y) = 1.36 + 4 \times 1.01 = 5.4.$$

(f) No formula can be used for solving the variance of XY directly, so we solve it by definition. Note that XY is also a random variable with its own pmf. For example, $\Pr(XY = 4) = 0.095$ has been found in Part (c). By repeating the calculations in Part (c) for other possible values of xy, we may construct the pmf of XY, as shown in Table 4. Let $S = \{0, 1, 2, 3, 4, 6, 8, 9, 12\}$ be the sample space of XY, it then follows that

$$\text{Var}(XY) = \sum_{xy \in S} (xy - \mathbb{E}[XY])^2 \Pr(XY = xy) \approx 8.58.$$

4. Let $\mu = \mathbb{E}[X]$ be the mean of X. We then have

$$\begin{aligned}
\text{Var}(X) \quad &\equiv \mathbb{E}[(X - \mu)^2] = \mathbb{E}[X^2 - 2X\mu + \mu^2] \quad &&(\text{By } (a+b)^2 = a^2 + 2ab + b^2) \\
&= \mathbb{E}[X^2] - \mathbb{E}[2X\mu] + \mathbb{E}[\mu^2] \quad &&(\text{Split the expectation}) \\
&= \mathbb{E}[X^2] - 2\mu\mathbb{E}[X] + \mu^2\mathbb{E}[1] \quad &&(\text{Take constants outside the expectations}) \\
&= \mathbb{E}[X^2] - 2\mu^2 + \mu^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2.
\end{aligned}$$

For Y in Problem 2, we have $\text{Var}(Y) = 1.01$, which is equal to $\mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = 3.9 - 1.7^2$.

5. (a) A discrete random variable.

   (b) A nonrandom variable.

   (c) A continuous random variable.

   (d) A continuous random variable.

   (e) A nonrandom variable.

   (f) A nonrandom variable.

   (g) A nonrandom variable.

   (h) A continuous random variable.

6. Let X be the number of voters favoring Jones out of the 100 voters, then X follows a Binomial distribution with 100 trials and probability 0.45. The probability that the survey suggests a victory for Jones is
$$\Pr(X > 50) = \Pr(X \geq 51) \approx 0.135.$$
In finding the numerical value of the probability, the MS Excel function BINOMDIST() may help.

7. (15 points; 5 points each) A manufacturer produces a product and sell it to a retailer. The manufacturer offers a menu of three contracts. In each contract, a number of products delivered and a payment are specified. The three contracts are $(200, \$20000)$, $(300, \$29500)$, and $(400, \$38500)$. For example, if the retailer chooses the first contract, she will pay $2000 to the manufacturer and obtain 20 units of that product. No matter how many products she obtains, the retailer will sell all of them to consumers. When she obtains 200 units, the retail price will be $120. For 300 units, $117. For 400 units, $114. The product may be defective with probability $p = 2\%$. Once a consumer purchases a defective product, the retailer must pay the consumer $200 as a penalty.

   (a) We first calculate the expected profit under contract 1. By selling 200 units at $120 per unit, the sales revenue is $120 \times 200 = \$24000$. Subtracting the procurement cost $20000 form it results in $4000 as the profit with no defect. Suppose X_1 is the number of defective products under contract 1, then because each unit will be defective with probability $p = 2\%$, X_1 follows a binomial distribution with 200 trials and probability 0.02. In expectation there will be $200 \times 0.02 = 4$ defective products, which result in a penalty of $\$200 \times 4 = \$800$. Subtracting with from $4000 leads to $3200, which is the expected profit under contract 1. The calculations for contracts 2 and 3 are shown in Table 5. According to the table, contract 3 should be chosen. The associated expected profit is $5500.

Contract	1	2	3
Sales revenue (\$)	24000	35100	45600
Profit with no defect (\$)	4000	5600	7100
Expected number of defects	4	6	8
Expected penalty (\$)	800	1200	1600
Expected profit (\$)	3200	4400	5500
Target (\$)	1500	2250	3000
Defects allowed	12.5	16.75	20.5
Desired probability	0.99977	0.99986	0.99992
Probability of paying the fine	0.00748	0.08184	0.28211
Expected fine (\$)	14.96	163.68	564.22
Expected profit (\$)	3185.04	4236.32	4935.78

Table 5: Calculations for Problem 7.

   (b) We first calculate the probability of achieving the target under contract 1. Because the number of products is 200, the target is $\$7.5 \times 200 = \$1500$. As the profit with no defect is $4000, the number of defects allowed is $\lfloor \frac{4000-1500}{200} \rfloor = 12$. The probability of achieving this goal is $\Pr(X \leq 12) \approx 0.99977$. The calculations for contracts 2 and 3 are shown in Table 5. According to the table, contract 3 should be chosen. The associated probability is 0.99992.

(c) For contract 1, the probability of being charged by the government is $\Pr(X \geq 10) \approx 0.00748$. This results in an expected fine $\$2000 \times 0.00748 \approx \$14.96$ and the expected profit $\$3200 - \$14.96 = \$3185.04$. The calculations for contracts 2 and 3 are shown in Table 5. According to the table, contract 3 should be chosen. The associated probability is $\$4935.78$.

8. (a) First, note that

$$\mathbb{E}[Z] = \sum_{z=1}^{\infty} z \Pr(Z = z) = \sum_{z=1}^{\infty} z p^{z-1}(1-p). \tag{1}$$

Multiply both sides by p, we obtain

$$p\mathbb{E}[Z] = \sum_{z=1}^{\infty} z p^{z}(1-p). \tag{2}$$

Subtracting (1) by (2), we obtain

$$(1-p)\mathbb{E}[Z] = \sum_{z=1}^{\infty} z p^{z-1}(1-p) - \sum_{z=1}^{\infty} z p^{z}(1-p)$$

$$= \left[1 \cdot (1-p) + 2 \cdot p(1-p) + 3 \cdot p^2(1-p) + \cdots \right]$$

$$- \left[1 \cdot p(1-p) + 2 \cdot p^2(1-p) + \cdots \right]$$

$$= (1-p) + p(1-p) + p^2(1-p) + \cdots = \frac{1-p}{1-p} = 1.$$

Therefore, $\mathbb{E}[Z] = \frac{1}{1-p}$.

(b) We have

$$\Pr(Z \leq z) = \sum_{t=1}^{z} t \Pr(Z = t) = (1-p) + p(1-p) + \cdots + p^{z-1}(1-p)$$

$$= (1-p)\left(1 + p + \cdots + p^{z-1}\right) = (1-p)\left(\frac{1-p^z}{1-p} \right) = 1 - p^z.$$

9. Let S_{XY}, S_X, and S_Y be the sample spaces of XY, X, and Y. We then have

$$\begin{aligned}
\mathbb{E}[XY] &= \sum_{xy \in S_{XY}} xy \Pr(XY = xy) \\
&= \sum_{x \in S_X} \sum_{y \in S_Y} xy \Pr(X = x, Y = y) \\
&= \sum_{x \in S_X} \sum_{y \in S_Y} xy \Pr(X = x) \Pr(Y = y) \quad \text{(because X and Y are independent)} \\
&= \sum_{x \in S_X} x \sum_{y \in S_Y} y \Pr(Y = y) \Pr(X = x) \quad \text{(taking x outside the summation over y)} \\
&= \sum_{x \in S_X} x \mathbb{E}[Y] \Pr(X = x) \quad \text{(By the definition of $\mathbb{E}[Y]$)} \\
&= \mathbb{E}[Y] \sum_{x \in S_X} x \Pr(X = x) = \mathbb{E}[X]\mathbb{E}[Y].
\end{aligned}$$

10. By the properties of expectations, we know

$$\mathbb{E}\left[(X - d)^2 \right] = \mathbb{E}[X^2] - 2d\mathbb{E}[X] + d^2 = \mathbb{E}[X^2] - 2d\mu + d^2.$$

Our goal is to find a minimizer d^* that solves

$$\min_{d \in \mathbb{R}} \mathbb{E}\left[X^2 \right] - 2d\mu + d^2.$$

Because this function is quadratic in d and d^2 has a positive coefficient, we know a stationary point attains the unique global minimum. As the first-order derivative of $\mathbb{E}[X^2] - 2d\mu + d^2$ with respect to d is $-2\mu + 2d$, setting it to zero results allows us to solve $-2\mu + 2d^* = 0$, which leads to $d^* = \mu$.

11. (a) Let X be the number of heads observed in ten trials, then X follows a binomial distribution with 10 trials and probability $p \in \{0.1, 0.2, ..., 0.9\}$. For each prior probability, we may calculate the probability of observing six heads in ten trials. For example, if $p = 0.9$, the probability of observing this is $\Pr(X = 6|10, 0.9) \approx 0.001$; if $p = 0.8$, the probability goes higher to $\Pr(X = 6|10, 0.8) \approx 0.01$. The nine probabilities associated with the nine priors are listed in the rows of "Possible values of p" of the column "$(10, 6)$" in Table 6. We may then find the posterior believes by dividing the probabilities by their sum (in the "Total" row). For example, the posterior believes on $p = 0.9$ is 0.012, on $p = 0.8$ is 0.097, etc.

Experiment setting	p	(n, x)		
		$(10, 6)$	$(100, 65)$	$(1000, 650)$
	0.9	0.001	0.00000	0.000000
	0.8	0.010	0.00002	0.000000
	0.7	0.022	0.00520	0.000009
	0.6	0.028	0.00546	0.000015
$\Pr(X = x|n, p)$	0.5	0.023	0.00010	0.000000
	0.4	0.012	0.00000	0.000000
	0.3	0.004	0.00000	0.000000
	0.2	0.001	0.00000	0.000000
	0.1	0.000	0.00000	0.000000
Total	–	0.101	0.01077	0.000024
	0.9	0.012	0.00000	0.000000
	0.8	0.097	0.00195	0.000000
	0.7	0.220	0.48244	0.379709
	0.6	0.276	0.50670	0.620291
Posterior believes	0.5	0.226	0.00891	0.000000
	0.4	0.123	0.00000	0.000000
	0.3	0.040	0.00000	0.000000
	0.2	0.006	0.00000	0.000000
	0.1	0.000	0.00000	0.000000

Table 6: Bayesian updating for Problem 11.

(b) The results are listed in the column "$(100, 65)$" of Table 6. Roughly speaking, we believe that it is only possible for p to be between 0.5 and 0.8.

(c) The results are listed in the column "$(1000, 650)$" of Table 6. Roughly speaking, we believe that it is only possible for p to be between 0.6 and 0.7.