Statistics I, Fall 2012
Suggested Solution for Homework 10

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. My name is Ling-Chieh Kung, but for this question each one of you should write down your name.

2. (a) The population mean of the exponential population is $\frac{1}{\lambda}$. We need to calculate the expected values of these estimators and check whether they are equal to $\frac{1}{\lambda}$. We have

$$\mathbb{E}\big[\hat{\lambda}_1\big] = \mathbb{E}[X_1] = \frac{1}{\lambda},$$
$$\mathbb{E}\big[\hat{\lambda}_2\big] = \mathbb{E}\left[\frac{X_1 + X_2}{2}\right] = \frac{\frac{1}{\lambda} + \frac{1}{\lambda}}{2} = \frac{1}{\lambda},$$
$$\mathbb{E}\big[\hat{\lambda}_3\big] = \mathbb{E}\left[\frac{X_1 + 2X_2 + 3X_3}{12}\right] = \frac{\frac{1}{\lambda} + \frac{2}{\lambda} + \frac{3}{\lambda}}{12} = \frac{1}{2\lambda},$$
$$\mathbb{E}\big[\hat{\lambda}_4\big] = \int_0^\infty e^x \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{(1-\lambda)x} dx = \frac{\lambda}{1-\lambda} e^{(1-\lambda)x}|_0^\infty = \frac{\lambda}{\lambda - 1}, \text{ and}$$
$$\mathbb{E}\big[\hat{\lambda}_5\big] = \mathbb{E}\left[\frac{X_1 + X_2 + X_3}{3}\right] = \frac{\frac{1}{\lambda} + \frac{1}{\lambda} + \frac{1}{\lambda}}{3} = \frac{1}{\lambda}.$$

Therefore, estimators $\hat{\lambda}_1$, $\hat{\lambda}_2$, and $\hat{\lambda}_5$ are unbiased.

(b) To answer this question, we need to calculate the variances of the three unbiased estimators. We have

$$\text{Var}\big(\hat{\lambda}_1\big) = \frac{1}{\lambda^2}, \quad \text{Var}\big(\hat{\lambda}_2\big) = \frac{1}{4}\left(\frac{2}{\lambda^2}\right) = \frac{1}{2\lambda^2}, \quad \text{andVar}\big(\hat{\lambda}_5\big) = \frac{1}{9}\left(\frac{3}{\lambda^2}\right) = \frac{1}{3\lambda^2}.$$

Therefore, $i = 5$ and $j = 1$. The efficiency of $\hat{\lambda}_5$ relative to $\hat{\lambda}_1$ is $\frac{1/3\lambda^2}{1/\lambda^2} = 3$.

3. (a) True. When the population size N becomes smaller, the finite population corrector $\sqrt{\frac{N-n}{N-1}}$ will become smaller. Therefore, the standard error will become smaller and the confidence interval will become smaller.

(b) True. A larger confidence level tends to enlarge the confidence interval. To maintain the same confidence interval, we will need to have a larger sample size.

(c) True. When one increase the sample size, the standard error will become smaller and thus the confidence interval will become smaller.

(d) False. It does not matter whether a normal distribution can approximate the binomial population or not. What matters is the distribution of the sample mean, which will be normal only if the sample size is larger than 30.

4. (a) Because the sample size is large and the population variance is known, we will use the z distribution to construct the interval. The three components we need are:

- The sample mean is 18.55.
- The standard error is $\frac{0.87}{\sqrt{120}} = 0.079$.
- The critical z value is $z_{0.025} = 1.96$ because the confidence level is 0.95.

Therefore, the interval bounds are $18.55 \pm 1.96 \times 0.079$, i.e., 18.39 and 18.7. With a 95% confidence level, the average number of pieces per package is between 18.39 and 18.7.

(b) Because the sample size is large and the population variance is known, we will use the z distribution to construct the interval. The three components we need are:

- The sample mean is 18.64.
- The standard error is $\frac{0.87}{\sqrt{240}} = 0.056$.
- The critical z value is $z_{0.025} = 1.96$ because the confidence level is 0.95.

Therefore, the interval bounds are $18.64 \pm 1.96 \times 0.056$, i.e., 18.53 and 18.75. With a 95% confidence level, the average number of pieces per package is between 18.53 and 18.75.

Note. Do you observe that increasing the sample size results in a smaller confidence interval?

5. (a) Because the sample size is small, we must have a normal population so that we may apply the z distribution. If this is not the case, a nonparametric method must be used.

(b) Because the population is normal and the population variance is known, we will use the z distribution to construct the interval. The three components we need are:

- The sample mean is 2.11.
- The standard error is $\frac{0.87}{\sqrt{240}}\sqrt{\frac{100-18}{100-1}} = 0.0236$. Note that we need to do the finite population correction because $n = 18 > 5 = 0.05N$.
- The critical z value is $z_{0.05} = 1.645$ because the confidence level is 0.9.

Therefore, the interval bounds are $2.11 \pm 1.645 \times 0.0236$, i.e., 2.072 and 2.157. With a 90% confidence level, the average price is between \$2.072 and \$2.157.

(c) Because the population is normal and the population variance is known, we will use the z distribution to construct the interval. The three components we need are:

- The sample mean is still 2.11.
- The standard error is still 0.0236.
- The critical z value is $z_{0.005} = 2.576$ because the confidence level is 0.99.

Therefore, the interval bounds are $2.11 \pm 2.576 \times 0.0236$, i.e., 2.048 and 2.181. With a 90% confidence level, the average price is between \$2.048 and \$2.181.

Note. Do you observe that increasing the confidence level results in a larger confidence interval?

(d) Based on our answer in Part (b), with probability 90% the average retail price is between \$2.072 and \$2.157. This implies that with probability 90% the average retail price is below \$2.18. Therefore, the wholesale price should be increased.

Note. If we make this decision based on the result in Part (c), the decision will become "No". Why? However, if we do a one-tail estimation with a 99% confidence level, the answer will go back to "Yes". Do you want to know why?