Statistics I, Fall 2012
Suggested Solution for Homework 13

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. (25 points; 5 points each) For the population mean μ, consider the following hypothesis

$$H_0: \mu = 7.5$$
$$H_a: \mu < 7.5.$$

with the sample mean $\bar{x} = 6.91$, sample size $n = 36$, population standard deviation $\sigma = 1.21$, and significance level $\alpha = 0.01$. Assume the population is normal.

(a) The probability of making a Type I error is $\alpha = 0.01$.

(b) We first need to find the rejection rule. For this hypothesis, we will reject H_0 if the critical value $\bar{x}^*$ satisfies

$$\Pr(\overline{X} < \bar{x}^*) = 0.01 \quad \Leftrightarrow \quad \bar{x}^* = 7.031.$$

The probability of making a Type II error is then calculated as

$$\beta = \Pr(\text{Accept } H_0 | H_0 \text{ is false}) = \Pr(\overline{X} > \bar{x}^* | \mu = 7.2)$$
$$= \Pr\left(Z > \frac{7.031 - 7.2}{1.21/\sqrt{36}}\right) = \Pr(Z > -0.8387) = 0.7992.$$

(c) In this case, the probability of making a Type II error is

$$\beta = \Pr(\text{Accept } H_0 | H_0 \text{ is false}) = \Pr(\overline{X} > \bar{x}^* | \mu = 7)$$
$$= \Pr\left(Z > \frac{7.031 - 7}{1.21/\sqrt{36}}\right) = \Pr(Z > 0.153) = 0.4392.$$

(d) In this case, we need to first calculate the new critical value $\bar{x}^{**}$, which satisfies

$$\Pr(\overline{X} < \bar{x}^*) = 0.05 \quad \Leftrightarrow \quad \bar{x}^* = 7.168.$$

The probability of making a Type II error is then calculated as

$$\beta = \Pr(\text{Accept } H_0 | H_0 \text{ is false}) = \Pr(\overline{X} > \bar{x}^{**} | \mu = 7)$$
$$= \Pr\left(Z > \frac{7.168 - 7}{1.21/\sqrt{36}}\right) = \Pr(Z > 0.8345) = 0.202.$$

(e) Depict the power of this test as a function of μ over the domain $[5.5, 7.5]$.
The power is depicted in Figure 1.

2. Let μ be the average weekly earning (in \$) of production workers today, $\overline{X}$ be the sample mean, and $\alpha = 0.05$ be the significance level. The hypothesis is

$$H_0: \mu = 424.2$$
$$H_0: \mu \neq 424.2.$$

Because the population variance is known and the sample size is large, we may use the z test. In this two-tailed test, because $432.7 > 424.4$, the p-value is

$$\Pr(\overline{X} > 432.7) = \Pr\left(Z > \frac{432.7 - 424.2}{32/\sqrt{70}}\right) = \Pr(Z > 2.2224) = 0.0131.$$

With a 5% significance level, ts the p-value is smaller than $\frac{\alpha}{2} = 0.025$, we reject H_0. There is a strong evidence showing that the mean weekly earning of a production worker has changed.

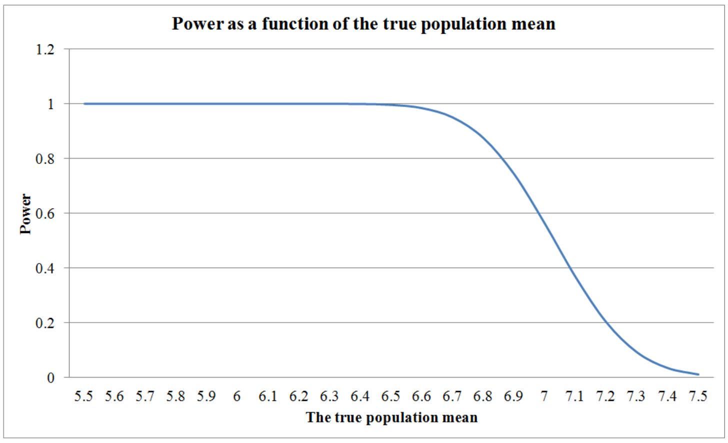

Figure 1: The power $1-\beta$ as a function of μ.

3. Let μ be the average age of a Canadian businesswoman today, $\overline{X}$ be the sample mean, and $\alpha = 0.01$ be the significance level. The hypothesis is

$$H_0 : \mu = 42$$
$$H_0 : \mu > 42.$$

Because the population variance is known and the sample size is large, we may use the z test. In this one-tailed test, the p-value is

$$\Pr(\overline{X} > 42) = \Pr\left(Z > \frac{44.4 - 42}{8.25/\sqrt{97}}\right) = \Pr(Z > 2.8651) = 0.0021.$$

With a 1% significance level, ts the p-value is smaller than $\alpha = 0.01$, we reject H_0. There is a strong evidence showing that the mean age of a Canadian businesswoman has increased.

4. Let μ be the average price of cleaning a 12' by 18' wall-to-wall carpet, $\overline{X}$ be the sample mean, and $\alpha = 0.1$ be the significance level. The hypothesis is

$$H_0 : \mu = 50$$
$$H_0 : \mu > 50.$$

Because the population variance is known and the population is normal, we may use the z test. The observed value is $\bar{x} = 52.36$. In this one-tailed test, the p-value is

$$\Pr(\overline{X} > 52.36) = \Pr\left(Z > \frac{52.36 - 50}{3.25/\sqrt{25}}\right) = \Pr(Z > 3.6308) = 0.0001.$$

With a 10% significance level, ts the p-value is smaller than $\alpha = 0.1$, we reject H_0. There is a strong evidence showing that the average price in the region in which the company operates is higher than \$50.

5. Let μ be the average diameter (in centimeters) of the punched hole, $\overline{X}$ be the sample mean, and $\alpha = 0.1$ be the significance level. The hypothesis is

$$H_0 : \mu = 1.9$$
$$H_0 : \mu \neq 1.9.$$

Because the population variance is unknown and the population is normal, we may use the t test. The observed sample mean and sample standard deviation are 1.864 and 0.0276. In this two-tailed test, because $1.864 < 1.9$ the p-value is

$$\Pr(\overline{X} < 1.864) = \Pr\left(T_9 < \frac{1.864 - 1.9}{0.0276/\sqrt{10}}\right) = \Pr(T_9 < -4.129) = 0.00128.$$

As the p-value is smaller than $\frac{\alpha}{2} = 0.05$, we reject H_0. With a 10% significance level, there is a strong evidence showing that the average diameter (in centimeters) of the punched hole is not 1.9 cm.

6. Let μ be the average rating today, $\overline{X}$ be the sample mean, and $\alpha = 0.1$ be the significance level. The hypothesis is

$$H_0 : \mu = 3.51$$
$$H_0 : \mu > 3.51.$$

Because the population variance is unknown and the population is normal, we may use the t test. (Note that for this problem we may also use the z test. Why?) The observed sample mean and sample standard deviation are 3.71 and 0.65. In this one-tailed test, the p-value is

$$\Pr(\overline{X} > 3.71) = \Pr\left(T_{63} > \frac{3.71 - 3.51}{0.65/\sqrt{64}}\right) = \Pr(T_{63} > 2.462) = 0.0083.$$

As the p-value is smaller than $\alpha = 0.01$, we reject H_0. With a 1% significance level, there is a strong evidence showing that the average rating has increased.

7. Let μ be the average yearly dental expenditure (in \$) per family, $\overline{X}$ be the sample mean, and $\alpha = 0.05$ be the significance level. The hypothesis is

$$H_0 : \mu = 1200$$
$$H_0 : \mu \neq 1200.$$

Because the population variance is unknown and the population is normal, we may use the t test. The observed sample mean and sample standard deviation are 1064.6 and 267.066. In this two-tailed test, because $1064.6 < 1200$, the p-value is

$$\Pr(\overline{X} < 1064.6) = \Pr\left(T_{24} < \frac{1064.6 - 1200}{267.066/\sqrt{25}}\right) = \Pr(T_{24} < -2.535) = 0.0091.$$

As the p-value is smaller than $\alpha = 0.01$, we reject H_0. With a 5% significance level, there is a strong evidence showing that the average yearly dental expenditure per family has increased.