Information Economics

Fall 2013

Suggested Solution for Homework 4

A1. (a) θ is the retailer's type, either r_H or r_L. $v(q)$ is the expected sales volume given the inventory level q, which is $\int_0^q x f(x)\,dx + \int_q^1 q f(x)\,dx$ $= q - \frac{1}{2}q^2$. Within $[0, 1]$, it is clear that $v'(q) > 0$ and $v''(q) < 0$.

(b) Our formula $\theta_i v'(q_i^{FB}) = c$ translates to $r_i(1 - q_i^{FB}) = c$, i.e., $q_i^{FB} = 1 - \frac{c}{r_i}$. The associated transfer is $t_i^{FB} = r_i v(q_i^{FB}) = \frac{1}{2r_i}(r_i^2 - c^2)$.

(c) $\max \ \beta(t_L - cq_L) + (1-\beta)(t_H - cq_H)$

s.t.
$$r_L(q_L - \tfrac{1}{2}q_L^2) - t_L \geqslant r_L(q_H - \tfrac{1}{2}q_H^2) - t_H \qquad \text{(IC-L)}$$
$$r_H(q_H - \tfrac{1}{2}q_H^2) - t_H \geqslant r_H(q_L - \tfrac{1}{2}q_L^2) - t_L \qquad \text{(IC-H)}$$
$$r_L(q_L - \tfrac{1}{2}q_L^2) - t_L \geqslant 0 \qquad \text{(IR-L)}$$
$$r_H(q_H - \tfrac{1}{2}q_H^2) - t_H \geqslant 0. \qquad \text{(IR-H)}$$

(d) Because $\frac{r_H - r_L}{r_H} = \frac{10-8}{10} = \frac{1}{5} < \frac{1}{2}$, we have

$r_H(1 - q_H^*) = c \iff q_H^* = 1 - \frac{2}{10} = \frac{4}{5}$ $\left(\text{by } \theta_H v'(q_H^*) = c\right)$ and

$r_L(1 - q_L^*) = c\left(\frac{1}{1 - \frac{1-\beta}{\beta}\frac{r_H - r_L}{r_L}}\right) \iff q_L^* = 1 - \frac{8/3}{8} = \frac{2}{3}$

$\left(\text{by } \theta_L v'(q_L^*) = c\left(\frac{1}{1 - \frac{1-\beta}{\beta}\frac{\theta_H - \theta_L}{\theta_L}}\right)\right)$.

The associated transfers are $t_L^* = 8\left(\frac{2}{3} - \frac{2}{9}\right) = \frac{32}{9}$

and $t_H^* = 10\left(\frac{12}{25} - \frac{4}{9}\right) + \frac{32}{9} = \frac{176}{45}$.

(e) For $q_L^* < q_H^*$, we have $\frac{2}{3} < \frac{4}{5}$.

For $q_L^* < q_L^{FB}$, we have $\frac{2}{3} < 1 - \frac{2}{8} = \frac{3}{4}$.

A2. (a) $\max \ \beta[v(q_L) - t_L] + (1-\beta)[v(q_H) - t_H]$

$$\text{s.t.} \quad t_H - \theta_H q_H \geq t_L - \theta_H q_L \qquad (\text{IC-H})$$
$$t_L - \theta_L q_L \geq t_H - \theta_L q_H \qquad (\text{IC-L})$$
$$t_H - \theta_H q_H \geq 0 \qquad (\text{IR-H})$$
$$t_L - \theta_L q_L \geq 0 \qquad (\text{IR-L})$$

(b) First, adding (IC-H) and (IC-L) shows that $q_L \geq q_H$ for all feasible solution. We then remove (IR-L) because it is redundant:

$t_L - \theta_L q_L \overset{(\text{IC-L})}{\geq} t_H - \theta_L q_H \overset{(\theta_L < \theta_H)}{\geq} t_H - \theta_H q_H \overset{(\text{IR-H})}{\geq} 0$. We then relax the (IC-H) constraint.

With (IC-L) and (IR-H) only, it is clear that $t_H = \theta_H q_H$ and $t_L = (\theta_H - \theta_L) q_H + \theta_L q_L$ because both constraints will be binding at any optimal solution. Replacing t_H and t_L in the objective function by $\theta_H q_H$ and $(\theta_H - \theta_L) q_H + \theta_L q_L$ leaves us the unconstrained problem

$$\max_{q_H, q_L} \ \beta[v(q_L) - (\theta_H - \theta_L) q_H - \theta_L q_L] + (1-\beta)[v(q_H) - \theta_H q_H].$$

The FOC implies $v'(q_L^*) = \theta_L$ and $v'(q_H^*) = \theta_H + \frac{\beta}{1-\beta}(\theta_H - \theta_L) > \theta_H > \theta_L$.

The optimal transfers are $t_H^* = \theta_H q_H^*$ and $t_L^* = (\theta_H - \theta_L) q_H^* + \theta_L q_L^*$.

To verify (IC-H), note that $t_H^* - \theta_H q_H^* = 0$ and $t_L^* - \theta_H q_L^*$

$= (\theta_H - \theta_L)(q_H^* - q_L^*) \leq 0$ because $q_H^* \leq q_L^*$.

(c) Monotonicity: $q_H^* \leq q_L^*$ because $v'(q_L^*) = \theta_L < \theta_H + \frac{\beta}{1-\beta}(\theta_H - \theta_L) = v'(q_H^*)$ and $v''(q) < 0$.

Efficiency at top: $q_L^* = q_L^{FB}$ because $v'(q_L^*) = \theta_L = v'(q_L^{FB})$.

No rent at bottom: $t_H^* - \theta_H q_H^* = 0$.

The "top" is the supplier with the low cost.

A3. Omitted.

B1. (a) Consider the full returns contract $(q, b, t) = (q_N^I, p, q_N^I p)$.

Because the manfacturer observes that the retailer does not forecast, his belief on demand is D_N and q_N^I maximizes system expected profit. Moreover, the retailer will accept the full returns contract with no rent. This implies that the manufacturer will offer the above full returns contract. The retailer earns 0 and the manufacturer earns $\Pi_N(q_N^I)$

(b) Again, consider the full returns contracts $(q_H, b_H, t_H) = (q_H^I, p, q_H^I p)$ and $(q_L, b_L, t_L) = (q_L^I, p, q_L^I p)$. We know this menu will give the manufacturer the highest possible profit as long as both types of retailer will choose the contract intended for her. It turns out that this is true, as the high-type consumer will earn 0 regardless of the contract she selects. Therefore, this is optimal. The manufacturer earns $\Pi_F(q_H^I, q_L^I)$ and the retailer earns 0. Note that her private information cannot protect her!

(c) As long as $k > 0$, the retailer should not forecast.

B2. In Pasternack (1985), offering full returns with full credits induces the retailer to order q such that $Pr(D \leq q) = 1$, where D is the demand. This means the contract is too generous. However, as in Taylor and Xiao (2009) the retailer will be forced to order q_S^I, $S \in \{N, H, L\}$, the system-optimal quantity, such a overstocking situation will not occur. In short, it is because that the retailer chooses the order quantity in Pasternack (1985) but cannot do so in Taylor and Xiao (2009), we see the difference.

B3 (a) Let $\pi(q,r) = p\left\{\int_{\mu-r}^{q} x\left(\frac{1}{2r}\right)dx + \int_{q}^{\mu+r} q\left(\frac{1}{2r}\right)dx\right\} - cq = \frac{p}{2r}\left[\frac{-1}{2}q^2 - \frac{1}{2}(\mu-r)^2 + q(\mu+r)\right] - cq$

and $\pi(r) = \max_q \pi(q,r) = \pi(q^*(r), r)$, where $q^*(r)$ is the optimal production quantity given r. First, we know $q^*(r)$ satisfies $1 - \frac{q^*(r)-(\mu-r)}{2r} = \frac{c}{p}$, so $q^*(r) = \mu - r + (2r)(1-\frac{c}{p})$.

Now, by the envelope theorem, $\frac{d}{dr}\pi(r) = \frac{\partial}{\partial r}\pi(q,r)\Big|_{q=q^*(r)}$

$= \frac{\partial}{\partial r}\left[\frac{p}{2r}\left(-\frac{1}{2}q^2 + q(\mu+r) + \frac{1}{2}(\mu-r)^2\right) - cq\right]\Big|_{q=q^*(r)}$

$= \left[-\frac{p}{2r^2}\left(-\frac{1}{2}q^2 + q(\mu+r) - \frac{1}{2}(\mu-r)^2\right) + \frac{p}{2r}\left(q + (\mu-r)\right)\right]\Big|_{q=q^*(r)}$

$= \frac{p}{4r^2}\left[q(q-2\mu) + (\mu-r)(\mu+r)\right]\Big|_{q=q^*(r)}$

$= \frac{p}{4r^2}\left[-\left[\mu-r+(2r)(1-\frac{c}{p})\right]\left[\mu+r-(2r)(1-\frac{c}{p})\right] + (\mu+r)(\mu-r)\right]$

$= \frac{p-c}{2r}\left[-(\mu+r) + (\mu-r) + 2r(1-\frac{c}{p})\right] = -\frac{c}{p}(p-c) < 0$. Therefore, improving the retailer's accuracy strictly benefits the system.

(b) Let $\pi_M(w,r) = (w-c)\,q^*(r) = (w-c)\left[\mu-r+2r(1-\frac{w}{p})\right] = (w-c)(\mu+r-\frac{2rw}{p})$ and $\pi_M(r) = \max_w \pi_M(w,r) = \pi_M(w^*(r), r)$, where $w^*(r)$ is the optimal wholesale price given r. First, we know $w^*(r) = \frac{p}{4r}\left(\mu+r+\frac{2rc}{p}\right) = \frac{p}{4r}(\mu+r) + \frac{c}{2}$. By the envelope theorem, $\frac{d}{dr}\pi_M(r) = \frac{\partial}{\partial r}\pi_M(w,r)\Big|_{w=w^*(r)} = (w-c)\left(1-\frac{2w}{p}\right)\Big|_{w=w^*(r)}$

$= \frac{1}{p}\left(w^*(r)-c\right)\left(p-2w^*(r)\right)$. For $w^*(r)-c = \frac{p}{4r}(\mu+r) - \frac{c}{2} = \frac{1}{4r}(p\mu+pr-2rc)$, because $\mu \geq r$ and $p > c$, $w^*(r)-c > 0$. For $p-2w^*(r) = \frac{1}{2r}(2pr-p\mu-pr-2rc)$

$= \frac{1}{2r}(pr-p\mu-2rc)$, because $\mu \geq r$, $p-2w^*(r) < 0$. Therefore, $\frac{d}{dr}\pi_M(r) < 0$ and improving the retailer's accuracy strictly benefits the manufacturer.