Information Economics

Fall 2013

Suggested Solution for Homework 5

1. We want to solve
$$\left\{\begin{array}{lll} \max & \mathbb{E}\left[(1-\beta(\theta))(\theta+\beta(\theta))-\alpha(\theta)\right] & \\ \text{s.t.} & CE_s(\theta) \geq CE_s(\theta,\tilde{\theta}) & \forall \theta, \tilde{\theta} \\ & CE_s(\theta) \geq 0 & \forall \theta, \end{array}\right\}$$
where $CE_s(\theta,\tilde{\theta}) = \alpha(\tilde{\theta}) + \beta(\tilde{\theta})\theta + \frac{1}{2}[\beta(\tilde{\theta})]^2(1-\rho\delta^2)$. First, note that $\frac{d}{d\theta}CE_s(\theta) = \frac{\partial}{\partial\theta}CE_s(\theta,\tilde{\theta})\Big|_{\tilde{\theta}=\theta} = \beta(\theta) \geq 0$, which implies $CE_s(\theta) = \int_{-\infty}^{\theta}\beta(x)\,dx + CE_s(-\infty)$. Because $\beta(\theta) \geq 0\ \forall\theta$, $CE_s(\theta)$ is nondecreasing in θ and thus $CE_s(-\infty)=0$ at any optimal solution. This also satisfies all the IR constraints. Because $CE_s(\theta) = \alpha(\theta) + \beta(\theta)\theta + \frac{1}{2}[\beta(\theta)]^2(1-\rho\delta^2) = \int_{-\infty}^{\theta}\beta(x)\,dx$, at any optimal solution $\alpha(\theta) = \int_{-\infty}^{\theta}\beta(x)\,dx - \left\{\beta(\theta)\theta + \frac{1}{2}[\beta(\theta)]^2(1-\rho\delta^2)\right\}$.

Ignoring the IC constraints for a while, the problem becomes
$$\max\ \mathbb{E}\left[\theta + \beta(\theta) - [\beta(\theta)]^2 + \tfrac{1}{2}[\beta(\theta)]^2(1-\rho\delta^2) - \int_{-\infty}^{\theta}\beta(x)\,dx\right]$$
$$= \max\ \mathbb{E}\left[\theta + \beta(\theta) - \tfrac{1}{2}[\beta(\theta)]^2(1+\rho\delta^2) - H(\theta)\beta(\theta)\right],$$
where the equality is due to
$$\mathbb{E}\left[\int_{-\infty}^{\theta}\beta(x)\,dx\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\theta}\beta(x)\,dx\, f(\theta)\,d\theta = \int_{-\infty}^{\theta}\beta(x)\,dx\, F(\theta)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}F(\theta)\beta(\theta)\,d\theta$$
$$= \int_{-\infty}^{\infty}[1-F(\theta)]\beta(\theta)\,d\theta = \mathbb{E}[H(\theta)\beta(\theta)].$$
By the FOC, the optimal $\beta(\theta)$ satisfies $1-\beta(\theta)(1+\rho\delta^2) - H(\theta) = 0$ if $1 > H(\theta)$ or 0 otherwise. Therefore, we have
$$\beta^*(\theta) = \frac{[1-H(\theta)]^+}{1+\rho\delta^2},$$
which is nondecreasing. To verify the IC constraints, it suffices to verify that the local IC constraints $\alpha'(\theta) + \beta'(\theta)\theta + \beta(\theta)\beta'(\theta)(1-\rho\delta^2) = 0$ are satisfied for all θ. To see this, note that $\frac{d}{d\theta}CE_s(\theta) = \beta(\theta)$ by the envelope theorem and $\frac{d}{d\theta}CE_s(\theta) = \alpha'(\theta) + \beta'(\theta)\theta + \beta(\theta) + \beta(\theta)\beta'(\theta)(1-\rho\delta^2)$ by direct differentiation. Direct comparisons show that the local IC constraints are satisfied. This, together with the monotonicity condition that $\beta^*(\theta)$ is nondecreasing, implies that the IC constraints are satisfied.

2. (a) Given a contract (w, t), the retailer's optimal order quantity is $q^* = 1 - \frac{w}{p}$.

The expected sales volume is $\int_0^{q^*} x\,dx + \int_{q^*}^{1} q\,dx = q^* - \frac{1}{2}(q^*)^2$. The manufacturer solves

$$\left\{\begin{array}{l} \max\limits_{w,t} \ (w-c)\left(1-\frac{w}{p}\right) + t \\ \text{s.t.} \ \ p\left(1-\frac{w}{p} - \frac{1}{2}\left(1-\frac{w}{p}\right)^2\right) - w\left(1-\frac{w}{p}\right) - t \geqslant 0, \end{array}\right\}$$

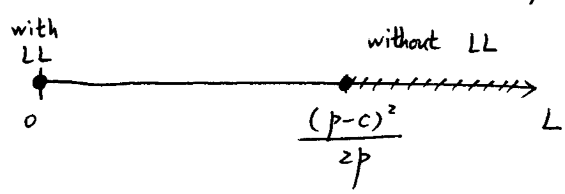
where the constraint is to ensure the retailer a nonnegative expected profit. At any optimal solution, the constraint will be binding and thus the problem reduces to

$\max\limits_{w} \ (p-c)\left(1-\frac{w}{p}\right) - \frac{1}{2}p\left(1-\frac{w}{p}\right)^2$. The FOC gives the optimal solution $w^* = c$.

The corresponding transfer is $t^* = \frac{(p-c)^2}{2p}$, which is the manufacturer's expected profit. In equilibrium, the order quantity is efficient and the retailer earns nothing in expectation. Therefore, the manufacturer earns the efficient system's expected profit.

(b) This is the standard indirect newsvendor problem that we have solved in the past. The equilibrium order quantity is $\frac{1}{2}\left(1-\frac{c}{p}\right)$ while the equilibrium wholesale price is $\frac{p+c}{2}$. With limited liability, the system is less efficient ($\frac{1}{2}\left(1-\frac{c}{p}\right)$ is inefficient), the retailer earns more (with a positive expected profit), and the manufacturer earns less. This is because the manufacturer now cannot use the fixed payment as a coordinating instrument. As double marginalization arises, the system becomes less efficient. The limited liability protects the retailer and hurts the manufacturer.

(c) Consider the figure



With the constraint $t \leq L$, if $L = 0$, this gives the situation described in Part (b). If $L \geqslant \frac{(p-c)^2}{2p}$, because the manufacturer only wants to charge $t^* = \frac{(p-c)^2}{2p}$, the constraint will not be violated and the equilibrium in Part (a) will still be the equilibrium.

3. (a) By observing the market condition θ, the manufacturer solves

$$\left\{\begin{array}{ll} \max\limits_{\alpha,\beta,a} & (1-\beta)\theta a - \alpha \\ \text{s.t.} & \alpha + \beta\theta a - \frac{1}{2}a^2 \geq 0 \\ & \alpha \geq 0,\ \beta \geq 0,\ a \geq 0. \end{array}\right\}$$

In particular, the constraint $\alpha \geq 0$ is required because the salesperson has limited liability. At an optimal solution, either $\alpha \geq 0$ or $\alpha + \beta\theta a - \frac{1}{2}a^2 \geq 0$ will be binding. If $\alpha = 0$, β will be adjusted to make $\beta\theta a = \frac{1}{2}a^2$, so the problem reduces to $\max\limits_{a}\ \theta a - \frac{1}{2}a^2$, which is optimized at $a^* = \theta$. It then follows that $\beta^* = \frac{1}{2}$ and the manufacturer's expected profit is $\frac{1}{2}\theta^2$.

If $\alpha = -\beta\theta a + \frac{1}{2}a^2$, the problem reduces to $\max\limits_{\beta, a}\ \theta a - \frac{1}{2}a^2$, which is equivalent to that with $\alpha = 0$. Therefore, a first-best equilibrium contract is $(\alpha^*, \beta^*, a^*) = (0, \frac{1}{2}, \theta)$. The manufacturer's ex ante expected profit is $\frac{1}{2}(\gamma\theta_L^2 + (1-\gamma)\theta_H^2)$.

(b) The manufacturer solves

$$\left\{\begin{array}{lll} \max & \gamma\big((1-\beta_L)\theta_L^2\beta_L - \alpha_L\big) + (1-\gamma)\big((1-\beta_H)\theta_H^2\beta_H - \alpha_H\big) & \\ \text{s.t.} & \alpha_L \geq 0,\ \alpha_H \geq 0,\ \beta_L \geq 0,\ \beta_H \geq 0, & \\ & \alpha_L + \frac{1}{2}\beta_L^2\theta_L^2 \geq \alpha_H + \frac{1}{2}\beta_H^2\theta_L^2 & \text{(IC-L)} \\ & \alpha_H + \frac{1}{2}\beta_H^2\theta_H^2 \geq \alpha_L + \frac{1}{2}\beta_L^2\theta_H^2 & \text{(IC-H)} \\ & \alpha_L + \frac{1}{2}\beta_L^2\theta_L^2 \geq 0 & \text{(IR-L)} \\ & \alpha_H + \frac{1}{2}\beta_H^2\theta_H^2 \geq 0 & \text{(IR-H)}. \end{array}\right\}$$

Note that the formulation comes from the anticipation that once (α, β) is chosen, the salesperson exerts effort $a^* = \beta\theta$ to maximize $\beta\theta a + \alpha - \frac{1}{2}a^2$. In this case, the salesperson earns $\alpha + \frac{1}{2}\beta^2\theta^2$. Now, to solve the manufacturer's problem, note that limited liability $(\alpha_L \geq 0,\ \alpha_H \geq 0)$ implies that both IR constraints are redundant.

If we ignore (IC-L) for a while, it is clear that $\alpha_L^* = 0$ at any optimal solution. Moreover, because $\beta_H \geq \beta_L$ (by adding the two IC constraints), (IC-H) now also becomes redundant, which further implies that $\alpha_H^* = 0$ at any optimal solution. It is then immediate that $\beta_H^* = \beta_L^* = \frac{1}{2}$. The expected effort level is $\frac{1}{2}(\gamma\theta_L + (1-\gamma)\theta_H)$ and the expected manufacturer's profit is $\frac{1}{4}(\gamma\theta_L^2 + (1-\gamma)\theta_H^2)$. Lastly, (IC-L) is satisfied.

3 (c) Given a contract (α, β), the salesperson's optimal effort is $a^* = \beta\theta$. With this in mind, the knowledgeable reseller solves $\left\{\begin{array}{ll} \max_{\alpha,\beta} & (v-\beta)\beta\theta^2 + u - \alpha \\ \text{s.t.} & \alpha \geq 0,\ \frac{1}{2}\beta^2\theta^2 + \alpha \geq 0. \end{array}\right\}$ As $\alpha \geq 0$,

the IR constraint is redundant. It then follows that $\alpha^* = 0$, $\beta^* = \frac{v}{2}$, $a^* = \frac{v}{2}\theta$, and the knowledgeable reseller's expected profit is $u + \frac{1}{4}v^2\theta^2$. At the manufacturer's contracting stage, the knowledgeable reseller's expected profit is $u + \frac{1}{4}v^2\left(\gamma\theta_L^2 + (1-\gamma)\theta_H^2\right)$ by accepting a contract (u, v). Therefore, the manufacturer solves

$$\left\{\begin{array}{ll} \max_{u,v} & \frac{1}{2}(1-v)v\left(\gamma\theta_L^2 + (1-\gamma)\theta_H^2\right) - u \\ \text{s.t.} & u + \frac{1}{4}v^2\left(\gamma\theta_L^2 + (1-\gamma)\theta_H^2\right) \geq 0. \end{array}\right\}$$

It is straightforward to show that the optimal solution consists of $v^* = 1$ and $u^* = -\frac{1}{4}\left(\gamma\theta_L^2 + (1-\gamma)\theta_H^2\right)$.

(d) The diligent reseller solves

$$\left\{\begin{array}{lll} \max & \gamma\left[(v-\beta_L)\theta_L a_L - \alpha_L\right] + (1-\gamma)\left[(v-\beta_H)\theta_H a_H - \alpha_H\right] + u & \\ \text{s.t.} & \alpha_L \geq 0,\ \alpha_H \geq 0 & \\ & \alpha_L + \beta_L\theta_L a_L - \frac{1}{2}a_L^2 \geq \alpha_H + \beta_H\theta_L a_H - \frac{1}{2}a_H^2 & (\text{IC-L}) \\ & \alpha_H + \beta_H\theta_H a_H - \frac{1}{2}a_H^2 \geq \alpha_L + \beta_L\theta_H a_L - \frac{1}{2}a_L^2 & (\text{IC-H}) \\ & \alpha_L + \beta_L\theta_L a_L - \frac{1}{2}a_L^2 \geq 0 & (\text{IR-L}) \\ & \alpha_H + \beta_H\theta_H a_H - \frac{1}{2}a_H^2 \geq 0 & (\text{IR-H}). \end{array}\right.$$

Here, note that the efficient contract $(\alpha_L^*, \beta_L^*, a_L^*, \alpha_H^*, \beta_H^*, a_H^*) = \left(\frac{1}{2}\theta_L^2, 0, \theta_L, \frac{1}{2}\theta_H^2, 0, \theta_H\right)$, which maximize the diligent reseller's expected profit, satisfies all constraints. It is thus the diligent reseller's optimal offer and gives her $(v-\frac{1}{2})\left(\gamma\theta_L^2 + (1-\gamma)\theta_H^2\right) + u$.

The manufacturer then solves $\left\{\begin{array}{ll} \max_{u,v} & \gamma\left[(1-v)\theta_L^2\right] + (1-\gamma)\left[(1-v)\theta_H^2\right] - u \\ \text{s.t.} & (v-\frac{1}{2})\left(\gamma\theta_L^2 + (1-\gamma)\theta_H^2\right) + u \geq 0. \end{array}\right\}$ $v^* = 1$

and $u^* = -\frac{1}{2}\left(\gamma\theta_L^2 + (1-\gamma)\theta_H^2\right)$ form an optimal solution.

(e)

Part	(a)	(b)	(c)	(d)
Manufacturer's expected profit	$\frac{1}{2}\mathbb{E}[\theta^2]$	$\frac{1}{4}\mathbb{E}[\theta^2]$	$\frac{1}{4}\mathbb{E}[\theta^2]$	$\frac{1}{2}\mathbb{E}[\theta^2]$
Expected sales effort	$\mathbb{E}[\theta]$	$\frac{1}{2}\mathbb{E}[\theta]$	$\frac{1}{2}\mathbb{E}[\theta]$	$\mathbb{E}[\theta]$