Information Economics, Fall 2014
Homework 1

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1 Rules

Note 1. For this homework, each team can have ***at most three*** students. Though it is not required, we encourage you to complete this homework with your future teammates for class problems.

Note 2. This homework is due ***5:00 pm, September 19, 2014***. Please submit a ***hard copy*** into the instructor's mail box at the first floor of the Management Building II. As each team only needs to submit one copy, please indicate the names and student IDs of all team members on the first page.

Note 3. ***All*** the students who want to enroll in this course ***must*** submit this homework. If one does not do that, she/he will fail the course if she/he insists to take it.

2 Problems

1. (40 points; 5 points each) Please answer the following questions.

   (a) Let $f(x) = 4x_1^4 + 2x_1x_2^2 - x_2^2 + 3$. Find the gradient $\nabla f(x)$ and Hessian $\nabla^2 f(x)$.

   (b) Let $f(x) = \ln(x^2 + 2)e^{2x}$. Find $\frac{d}{dx} f(x)$.

   (c) Let $f(x) = x_1x_2^2 + e^{2x_1}$. Find $\int f(x)dx_1$ (you may ignore the constant).

   (d) Find $\frac{d}{dx}\int_0^x (t^3 + 3t - 2)dt$.

   (e) Let X be the outcome of rolling an unfair dice whose probability distribution is summarized in the following table:

x	1	2	3	4	5	6
$\Pr(X = x)$	0.1	0.1	0.2	0.2	0.3	0.1

   Find the expected value and variance of X.

   (f) Let $f(x) = kx^2$ be the probability density of a continuous random variable $X \in [0, 2]$. Find the value of k. Then find $\mathbb{E}[X]$.

   (g) Is $f(x) = x^3 + 2x^2$ a convex function over $[0, \infty)$? Prove it mathematically.

   (h) Over what region is $g(x) = x^3 - 2x^2$ a convex function? Prove it mathematically.

2. (20 points; 5 points each) Consider the following nonlinear program

$$\begin{aligned} z^* = \max \quad & x_1 - x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 4 \\ & -x_1^2 - (x_2 + 2)^2 \leq -4. \end{aligned}$$

   (a) Draw the feasible region. Is it a convex set?

   (b) Graphically find an optimal solution. You do not need to prove that it is indeed optimal.

   (c) Is there any local maximum that is not a global maximum? If so, find them.

   (d) Is there any point that satisfies the unconstrained FONC?

3. (10 points) Let $f(\cdot)$ and $g(\cdot)$ be two convex functions defined over the same convex domain F. Prove or disprove that $h(x) \equiv \max\{f(x), g(x)\}$ is a convex function over F.

   Note. Be aware that these functions may be non-differentiable. If they are both differentiable, does the problem becomes easier?

4. (10 points) Solve the following single-variate optimization problems.

   (a) (3 points) Find
   $$\operatorname*{argmin}_{x \in \mathbb{R}} \left\{ x^4 + 2x^3 + 1 \middle| x \in [-2, -1] \right\}.$$

   (b) (2 points) Find
   $$\operatorname*{argmax}_{x \in \mathbb{R}} \left\{ x^4 + 2x^3 + 1 \middle| x \in [-2, 0] \right\}.$$

   (c) (5 points) For a general nonlinear program, if the objective function is not convex over the feasible region, a way to solve the problem is to (1) find all points satisfying the FONC and then (2) compare all these points as well as all boundary points of the feasible region. A winner of this selection process will be a global optimum. With this in mind, find
   $$\operatorname*{argmin}_{x \in \mathbb{R}} \left\{ x^4 + 2x^3 + 1 \middle| x \in [-2, 1] \right\}.$$

   Note 1. The three feasible regions are all different!

   Note 2. The strategy mentioned above is not practical in general. Why?

5. (20 points; 5 points each) In a market, there is a monopolist selling one product. After the seller produces q unit of that product, these products will all be sold at the market-clearing price $a - bq$. The unit production cost of the product is c. The exogenous parameters a, b, and c are all known and positive.

   (a) Formulate the seller's problem for maximizing its total profit.

   Note. This should be a constrained program.

   (b) Is the seller's problem a convex program? Why or why not?

   (c) Solve the program to find the optimal production quantity q^* as a function of a, b, and c.

   (d) How does q^* change when a, b, or c changes? Provide economic intuitions to these mathematical results.

   Note. When you change one parameter, you may fix the other two parameters.