Operations Research, Spring 2013
Midterm Exam

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

Name: ____________________ Student ID: ____________________

Note. In total there are 110 points for this exam. If you get more than 100 points, your official score for this exam will only be 100.

1. (10 points; 2 points each) For the following statements, CIRCLE or BOX "T" if one is true or "F" if it is false. DO NOT provide any explanation.

   (a) T F If a solution for a standard form LP is an optimal solution, then it must be a basic feasible solution for this LP.

   (b) T F If a linear program has two distinct optimal solutions, then it has infinitely many optimal solutions.

   (c) T F If a primal linear program is unbounded, its dual program can only be infeasible.

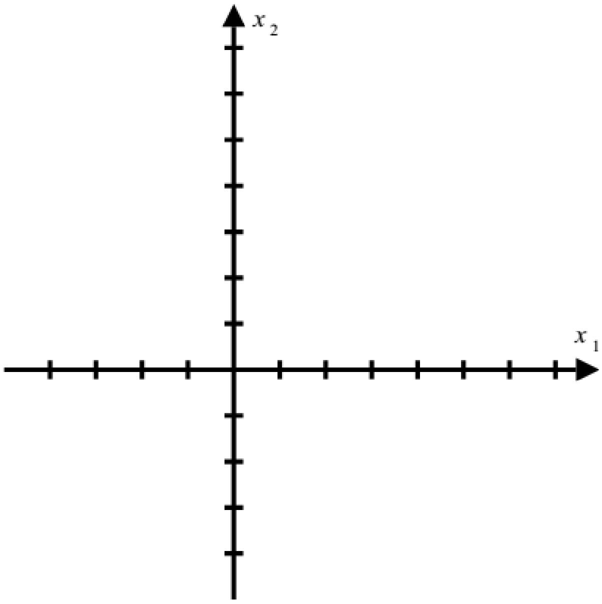
   (d) T F Consider a pair of primal and dual solutions and the strong duality theorem. If the two solutions have the same objective value, they will be primal and dual optimal, respectively.

   (e) T F Suppose a given LP (P) is unbounded and (D) is the dual of (P). Then the dual of (D) may be infeasible.

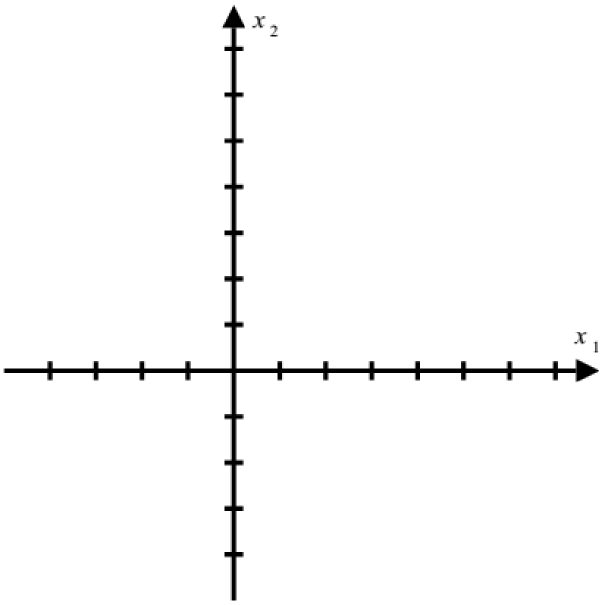
2. (15 points) Consider the following linear program

$$\begin{array}{rrcrcrl} z^* = \min & x_1 & + & x_2 & & & \\ \text{s.t.} & x_1 & - & 2x_2 & \leq & 4 & (1) \\ & 2x_1 & + & x_2 & \geq & -2 & (2) \\ & x_1 & - & x_2 & \geq & -3. & (3) \end{array}$$

(a) (10 points) Graphically solve this LP. You need to draw all the constraints, shade the feasible region, draw one isocost line, indicate the improving direction, and write down an optimal solution.



(b) (5 points) Suppose the objective function becomes "max $x_1 - x_2$". With the new objective function, solve the new LP graphically.



3. (10 points) Consider the following linear program

$$
(P_1) \quad \begin{array}{lrcrcl}
\max & x_1 & - & x_2 & & \\
\text{s.t.} & 2x_1 & - & x_2 & \geq & 4 \\
 & x_1 & - & x_2 & \leq & 3 \\
 & x_1 & & & \geq & 0 \\
 & & & x_2 & \leq & 0.
\end{array}
$$

(a) (3 points) Write down the standard form of the program (DO NOT add any artificial variable).

(b) (4 points) Find all the basic solutions of your standard form LP. For each basic solution, determine whether it is a basic feasible solution or not.

(c) (3 points) After some simplex iterations, you reach the following tableau:

$$
\begin{array}{cccc|c}
0 & 0 & 0 & 1 & 3 \\
\hline
1 & 0 & -1 & -1 & 1 \\
0 & 1 & 1 & 2 & 2
\end{array}
$$

Determine whether the *original program* (P_1) is infeasible, unbounded, having a unique optimal solution, or having multiple optimal solutions. If it has a unique solution, write it down. If it has multiple optimal solutions, write down two of them. In both cases, write down your solution based on the *original variables* x_1 and x_2.

4. (15 points) Consider the following LP

$$
\begin{array}{llllllll}
\max & x_1 & - & 2x_2 & & & & \\
\text{s.t.} & x_1 & & & - & 2x_3 & \leq & 6 \\
 & x_1 & - & 3x_2 & + & x_3 & \leq & 12 \\
 & & & 3x_2 & - & 2x_3 & \leq & 8 \\
 & & & & & x_i & \geq & 0 \quad \forall\, i = 1, ..., 3.
\end{array}
$$

Use the simplex method to solve the linear program. Whenever there are multiple choices of the entering or leaving variables, use the smallest index rule: Choose the variable with the smallest index. If the LP is infeasible or unbounded, clearly indicate why. If it has a unique solution, write it down. If it has multiple optimal solutions, write down only one of them but indicate the evidence of multiple optimal solutions.

5. (15 points) Consider the following LP

$$\begin{aligned} \max \quad & x_1 \\ \text{s.t.} \quad & x_1 - x_2 = 2 \\ & x_1 \leq 4 \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Use the two-phase implementation to run the simplex method to determine the feasibility of this program. If it is feasible, find one optimal solution in the *original variables* x_1 and x_2 or show that it is unbounded. Otherwise, show that it is infeasible. Whenever there are multiple choices of the entering or leaving variables, use the smallest index rule: Choose the variable with the smallest index.

6. (10 points) A company uses two items to produces three products. One unit of product 1 requires two units of item 1 and one unit of item 2. One unit of product 2 requires three units of item 1 and no item 2. One unit of product 3 requires one unit of item 1 and two units of item 2. Each unit of products 1, 2, and 3 can be sold at \$20, \$30, and \$15, respectively. The demands for products 1, 2, and 3 are at most 100, 50, and 120 units. Only 500 units of item 1 and 500 units of item 2 are available. The company can also bundle products 1 and 2 and sell the bundle of one unit of product 1 and one unit of product 2 at \$40 (note that one unit of bundle generates \$40, not \$80). The bundle is very popular and has no demand limitation. Formulate an LP that can maximize the company's sales revenue.

7. (10 points) Consider the following integer program

$$\begin{array}{rrcrcl} \max & 8x_1 & + & 7x_2 & & \\ \text{s.t.} & 4x_1 & + & 2x_2 & \leq & 28 \\ & 2x_1 & + & 3x_2 & \leq & 21 \\ & & x_i \in \mathbb{Z}_+ & \forall\, i = 1, 2. & & \text{(i.e., they are nonnegative integers)} \end{array}$$

Use the branch-and-bound algorithm to solve the program. When there are more than one variable having fractional values, branch on x_1. When there are multiple nodes to branch, branch the node with the highest objective value. Draw the complete branch-and-bound tree. For each branch, write down the constraint added. For each node, unless it is infeasible, write down its optimal solution and the corresponding objective value. You DO NOT need to show how you solve the subproblem. Finally, clearly indicate one optimal solution and its objective value.

8. (15 points) You are selling a product to a region with 10 counties, each contains 5 cities. For each county i ($i = 1, 2, ..., 10$), the construction cost of building a distributing center (DC) is H_i and the DC can serve at most K_i consumers. For each city j ($j = 1, 2, ..., 5$) in county i, the number of consumers is D_{ij} (i.e., you may serve at most D_{ij} consumers in this city), the cost of serving one consumer is C_{ij}, the revenue of serving one consumer is R_{ij}, and the construction cost of a retail store is F_{ij}. You may build at most one DC in each county, build at most one store in each city, and serve a consumer in a city only if you build a store in that city and a DC in the associated county. Note that you are NOT required to serve all consumers in a city when you build a store there.

(a) (10 points) Formulate an integer program that can maximize the total profit.

(b) (2 points) Write an additional constraint so that you will build at least three stores among the ten cities of counties 3 and 4.

(c) (3 points) Write additional constraints (and define new variables if you need) so that in at least five counties you build at least three stores in each of these five counties.

9. (10 points) Consider the following primal LP

$$\begin{array}{rlcccccccc} z^* = & \max & 3x_1 & + & 2x_2 & + & x_3 & & \\ & \text{s.t.} & x_1 & + & x_2 & + & 2x_3 & \geq & 6 \\ & & 2x_1 & + & 2x_2 & + & x_3 & \leq & 9 \\ & & & & x_i \geq 0 & & \forall\, i = 1, ..., 3. & & \end{array}$$

You are given the optimal tableau

$$\begin{array}{ccccc|c} 0 & 1 & 0 & \frac{1}{3} & \frac{5}{3} & 13 \\ \hline 0 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & 1 \\ 1 & 1 & 0 & \frac{1}{3} & \frac{2}{3} & 4 \end{array}$$

The optimal solution for the original problem $x^* = (x_1^*, x_2^*, x_3^*) = (4, 0, 1)$ and the corresponding objective value $z^* = 13$.

(a) (3 points) Find the shadow prices for the two constraints.

(b) (3 points) Find the dual LP of the primal LP.

(c) (4 points) DO NOT solve your dual LP graphically or using the simplex method. Instead, according to your primal optimal solution x^*, find a set of dual constraints that must be binding at any dual optimal solution. Then find a dual optimal solution by solving the resulting linear system. You may want to use your dual optimal solution to verify your solution in Part (a).