Operations Research, Spring 2013
Suggested Solution for Homework 02

Instructor: Ling-Chieh Kung
Department of Information Management
National Taiwan University

1. (a) Let x_1 and x_2 be two points in $S_1 \cap S_2$, then naturally x_1 and x_2 are both in S_1. As S_1 is convex, we have $\lambda x_1 + (1-\lambda)x_2 \in S_1$ for all $\lambda \in [0,1]$. Similarly, as S_2 is convex, we have $\lambda x_1 + (1-\lambda)x_2 \in S_2$ for all $\lambda \in [0,1]$. In then follows that, for any $\lambda \in [0,1]$, we have $\lambda x_1 + (1-\lambda)x_2 \in S_1 \cap S_2$. This implies that $S_1 \cap S_2$ is convex.

   (b) Let $S_1 = [0,1]$ and $S_2 = [2,3]$. We may pick $x_1 = 1$ and $x_2 = 2$ and then find that $\frac{1}{2}x_1 + \frac{1}{2}x_2 \notin S_1 \cup S_2$. Therefore, $S_1 \cup S_2$ is not convex.

2. (a) The feasible region and two isocost lines are illustrate in Figure 1.

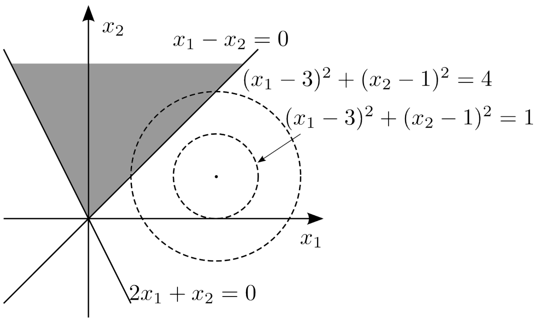

Figure 1: Graphical solution for Problem 2a.

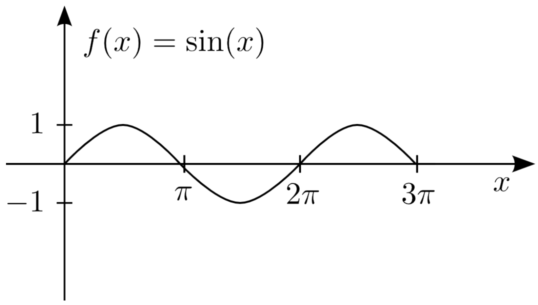

Figure 2: Graphical solution for Problem 3.

   (b) It is clear that we are looking for the feasible point that is closest to the point $(3,1)$. Graphically it can be found that the point $(2,2)$ is the closest feasible point, and thus $(2,2)$ is the optimal solution.

   (c) The only extreme point of the feasible region is $(0,0)$. As it is not an optimal solution, there is no extreme point optimal solution. There needs not to have an extreme point optimal solution because this is not a linear program: The objective function is nonlinear.

3. The function is depicted in Figure **??**.

   (a) There are three local minima: 0, $\frac{3}{2}\pi$, and 3π. The global minimum is $\frac{3}{2}\pi$.

   (b) There are two local maxima: $\frac{\pi}{2}$, $\frac{5}{2}\pi$. Both of them are global maxima.

4. (a) Let

$$x_1 = \text{hours of running process 1 and}$$
$$x_2 = \text{hours of running process 2}$$

   be the decision variables. Then the problem can be formulated as

$$\begin{array}{lrcrcll}
\min & 4x_1 & + & x_2 & & & \\
\text{s.t.} & 3x_1 & + & x_2 & \geq & 10 & \text{(Requirement of chemical A)} \\
 & x_1 & + & x_2 & \geq & 5 & \text{(Requirement of chemical B)} \\
 & x_1 & & & \geq & 3 & \text{(Requirement of chemical C)} \\
 & x_1 & & & \geq & 0 & \text{(This is not required since we already have } x_1 \geq 3\text{)} \\
 & & & x_2 & \geq & 0. &
\end{array}$$

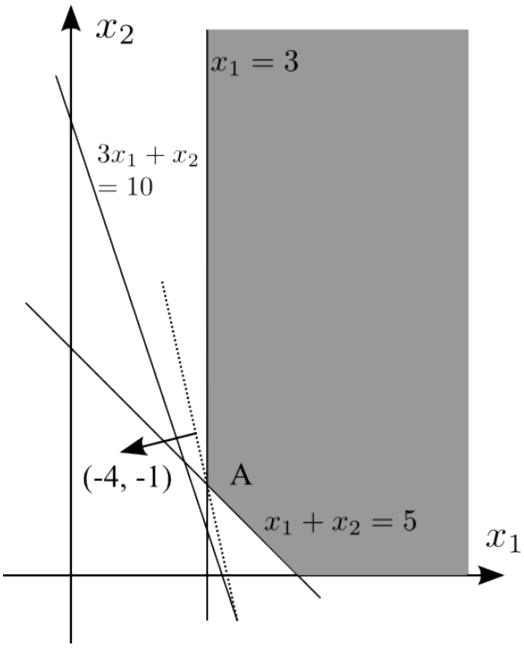

Figure 3: Graphical solution for Problem 4a.

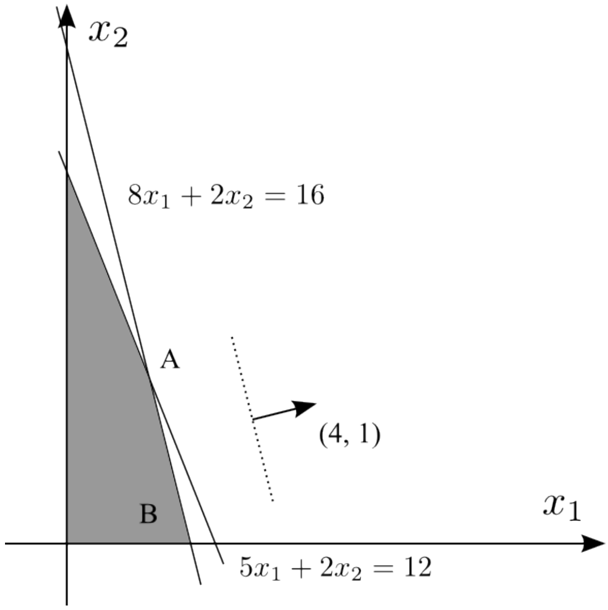

Figure 4: Graphical solution for Problem 5a.

(b) The graphical solution of this problem is shown in Figure 3. The feasible region is the shaded zone. The dotted line is the isocost line with $(-4, -1)$ as the improving direction. The optimal solution is point A, which is $(3, 2)$. The minimized objective value is $4 \times 3 + 2 = 14$. Therefore, Leary Chemical should run 3 hours of process 1 and 2 hours of process 2. The company will have \$14 as its cost.

5. (a) The graphical solution of this problem is shown in Figure 4. The feasible region is the shaded zone. The dotted line is the isoprofit line with $(4, 1)$ as the improving direction. As shown in this figure, all points between $A = (\frac{4}{3}, \frac{8}{3})$ and $B = (2, 0)$ are optimal. Therefore, this problem has multiple optimal solutions.

   (b) The binding constraints at the point $(x_1, x_2) = (2, 0)$ are $8x_1 + 2x_2 \leq 16$ and $x_2 \geq 0$.

   (c) There is no binding constraints at the point $(x_1, x_2) = (1, 3)$.

6. (a) True. An LP is unbounded implies that we can push its isoprofit line as far as we want and still have feasible solution at the line. This implies the feasible region is unbounded.

   (b) False. For example, consider the LP of maximizing x_1 subject to $x_1 \leq 10$. For this LP, the feasible region is unbounded (all real numbers that is no greater than 10 are feasible), but there is an optimal solution: $x_1^* = 10$ optimizes the LP.

7. The graphical solution of this problem is shown in Figure 5. For each constraint, we associate an arrow indicating the feasible side. From the figure we can see that there is no feasible solution for this problem and thus this problem is infeasible.

   Another way to see the infeasibility of this problem is to multiply the third constraint by -1. As it becomes $x_1 - x_2 \leq -3$, it is clear that we can not find a pair of x_1 and x_2 such that $x_1 - x_2$ is no less than 0 and no greater than -3 at the same time. Therefore, this problem is infeasible.

8. To formulate this problem, we label the four shifts as in the following table.

Shift Number	1	2	3	4
Time	0 - 6	6 - 12	12 - 18	18 - 24

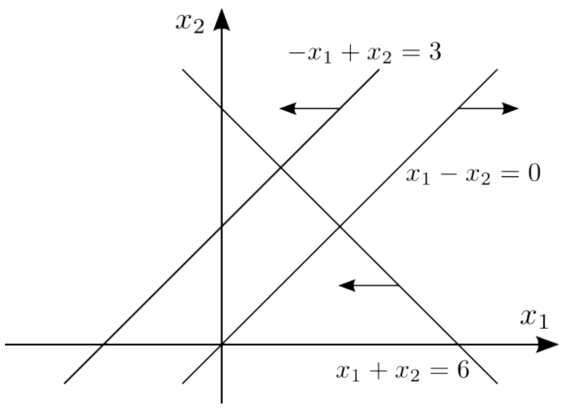

Figure 5: Graphical solution for Problem 7.

The decision variables are

$$x_{ij} = \text{number of officers working at shifts } i \text{ and } j,\ i = 1, ..., 4,\ j = i+1, ..., 4.$$

The problem can then be formulated as[1]

$$\begin{array}{lll}
\min & 144(x_{12} + x_{23} + x_{34} + x_{14}) + 216(x_{13} + x_{24}) & \\
\text{s.t.} & x_{12} + x_{13} + x_{14} \geq 15 & \text{(Demand in shift 1)} \\
& x_{12} + x_{23} + x_{24} \geq 5 & \text{(Demand in shift 2)} \\
& x_{13} + x_{23} + x_{34} \geq 12 & \text{(Demand in shift 3)} \\
& x_{14} + x_{24} + x_{34} \geq 6 & \text{(Demand in shift 4)} \\
& x_i \geq 0 \quad \forall\ i = 1, ..., 4. &
\end{array}$$

Here $144 = 12 \times 12$ is the per worker wage for one who works in two consecutive shifts. Similarly, $216 = 18 \times 12$ is that for one who works in two nonconsecutive shifts. The objective function minimize the total payments while the constraints guarantee that there are enough workers for every shifts.[2]

9. First, we label the two refinery at Los Angeles and Chicago as refinery 1 and 2 and the two distribution points at Houston and New York City as distribution points 1 and 2. Then let the decision variables be

$$\begin{aligned}
x_i &= \text{million barrels of capacity created for refinery } i,\ i = 1, 2, \text{ and} \\
y_{ij} &= \text{million barrels of oil shipped from refinery } i \text{ to distribution point } j,\ i = 1, 2,\ j = 1, 2.
\end{aligned}$$

For parameters, let P_{ij} be the profit (in thousands) per million barrels of oil shipped from refinery i to distribution point j, C_i be the unit cost (in thousands) of expanding capacity for one million barrel in refinery i, K_i be the current capacity (in million barrel) in refinery i, and D_j be the demand size (in million barrels) at distribution point j for all $i = 1, 2$ and $j = 1, 2$:

$$P = \begin{bmatrix} 20 & 15 \\ 18 & 17 \end{bmatrix}, \quad C = \begin{bmatrix} 120 \\ 150 \end{bmatrix}, \quad K = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 5 \end{bmatrix}.$$

[1] It does not matter if you use 12 and 18 instead of 144 and 216 in the formulation. The optimal solution will not be affected. However, using 144 and 216 is suggested because by doing so the objective function gives the amount of total payments as we desire.

[2] We can still write down the compact formulation for this problem. We do not include it here because it is a little bit too complicated. It would be great if you try to work it out by yourself. In case you have any question, I'm more than happy to discuss with you.

With the definitions of variables and parameters, we formulate the problem as

$$
\begin{aligned}
\max \quad & 10\sum_{i=1}^{2}\sum_{j=1}^{2} P_{ij}y_{ij} - \sum_{i=1}^{2} C_i x_i \\
\text{s.t.} \quad & \sum_{i=1}^{2} y_{ij} \leq D_j \quad \forall\, j = 1,2 \\
& \sum_{j=1}^{2} y_{ij} \leq K_i + x_i \quad \forall\, i = 1,2 \\
& x_i, y_{ij} \geq 0 \quad \forall\, i = 1,2, j = 1,2.
\end{aligned}
$$

The objective function consists of two parts, the 10-year total profit and the one-time expansion cost. The first constraint ensures that the total sales at each distribution point is at most the demand size. The second constraint ensures that the total production quantity at each refinery does not excess the (post-expansion) capacity. The last constraint is the nonnegativity constraint.