Operations Research, Spring 2013
Suggested Solution for Homework 03

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1. (a) True. The maximum number of distinct basic solutions is the number of distinct bases.
   (b) True. The number of distinct bases is $\binom{n}{m}$.
   (c) False. The number of distinct basic feasible solution is smaller than $\binom{n}{m}$ for a degenerate LP.
   (d) True. The number of distinct bases is exactly $\binom{n}{m}$.
   (e) False. A basic feasible solution has exactly $m(n-m)$ adjacent basic solutions. Some of them may be infeasible.
   (f) False. Some of them may be infeasible.
   (g) True. Some of the adjacent basic solutions may be infeasible.
   (h) False. For an unbounded LP, there may be a basic feasible solution which has no better adjacent basic feasible solution but that basic feasible solution is still not optimal.
   (i) False. If an LP is finitely optimal, what we can make sure is that there exist an optimal basic feasible solution. However, not all optimal solutions are basic solutions.
   (j) False. If an LP is finitely optimal, what we can make sure is that there exist an optimal basic feasible solution. However, not all optimal solutions are basic solutions.

2. The standard form is

$$
\begin{aligned}
\min\quad & 3x_1 + x_2 \\
\text{s.t.}\quad & x_1 - x_3 = 3 \\
& x_1 + x_2 + x_4 = 4 \\
& 2x_1 - x_2 = 3 \\
& x_j \geq 0 \quad \forall j = 1, ..., 4.
\end{aligned}
$$

3. (a) The standard form is

$$
\begin{aligned}
\max\quad & 3x_1 + 2x_2 \\
\text{s.t.}\quad & 2x_1 + x_2 + x_3 = 100 \\
& x_1 + x_2 + x_4 = 80 \\
& x_1 + x_5 = 40 \\
& x_j \geq 0 \quad \forall j = 1, ..., 5,
\end{aligned}
$$

   where x_3, x_4, and x_5 are slack variables.

   (b) Since in the standard form we have five variables and three constraints, there should be three basic variables and two nonbasic variables in a basic solution. The ten possible ways to choose two (nonbasic) variables to be 0 are listed in the table below.

x_1	x_2	x_3	x_4	x_5	Basic feasible solution?
0	0	100	80	40	Yes
0	100	0	-20	40	No
0	80	20	0	40	Yes
0	N/S	N/S	N/S	0	No
50	0	0	30	-10	No
80	0	-60	0	-40	No
40	0	20	40	0	Yes
20	60	0	0	20	Yes
40	20	0	20	0	Yes
40	40	-20	0	0	No

For each possibility, we try to solve the remaining three basic variables. Note that when x_1 and x_5 are chosen to be nonbasic, we can not find any solution that satisfies constraint $x_1 + x_5 = 40$ (the entries "N/S" means "no solution"). This happens because the constraint $x_1 \leq 40$ is parallel to the constraint $x_1 \geq 0$. The five basic feasible solutions are indicated in the last column.

4. Since in the standard form we have four variables and two constraints, there should be two basic variables and two nonbasic variables in a basic solution. The six possible ways to choose two (nonbasic) variables to be 0 are listed in the table below. For each possibility, we try to solve the remaining two basic variables. The last column indicates the corresponding extreme points of the feasible region shown in Figure 1, if the basic solution is feasible.

x_1	x_2	x_3	x_4	Extreme Point
0	0	-28	-24	(Infeasible)
0	14	0	144	B
0	2	-24	0	(Infeasible)
4	0	0	-16	(Infeasible)
12	0	56	0	C
3.6	1.4	0	0	E

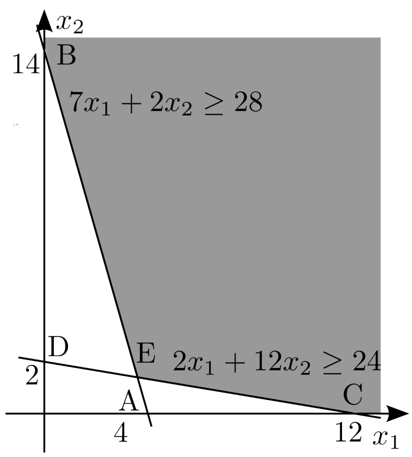

Figure 1: Extreme points for Problem 4

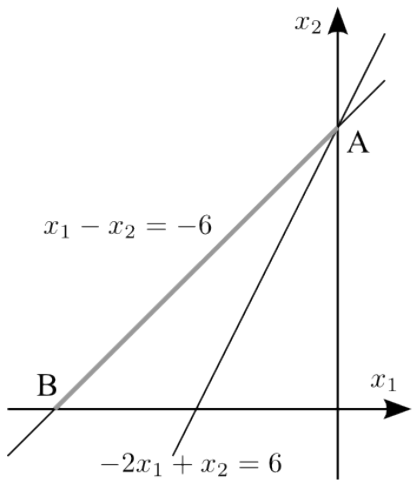

Figure 2: Extreme points for Problem 5

5. (a) The standard form is

$$\begin{aligned} \min \quad & x_1 + x_2 \\ \text{s.t.} \quad & 2x_1 + x_2 - x_3 = 6 \\ & x_1 + x_2 = 6 \qquad\qquad (\text{or } -x_1 - x_2 = -6) \\ & x_1 \geq 0, x_2 \geq 0. \end{aligned}$$

Note that we replace $-x_1$ by x_1 because all variables must be nonnegative in a standard form linear program.

(b) Since in the standard form we have three variables and two constraints, there should be two basic variables and one nonbasic variables in a basic solution. The three possible ways to choose two (nonbasic) variables to be 0 are listed in the table below.

x_1	x_2	x_3	Extreme Point
0	6	0	A
6	0	6	B
0	6	0	A

For each possibility, we try to solve the remaining two basic variables. Note that when we set $x_1 = 0$, we get $x_3 = 0$ when we solve the remaining two by two system. Similarly, when we set $x_3 = 0$, we get $x_1 = 0$. Therefore, there are only *two* basic solutions for this LP. The last column indicates the corresponding extreme points of the feasible region according to Figure 2, if the basic solution is feasible.

6. (a) The mapping of constraints and variables is demonstrated in the table below. It is clear that slack variables are associated with functional constraints and original variables are associated with nonnegativity constraints.

Constraint	Constraint type	Variable	Variable type
$2x_1 - x_2 \leq 4$	Functional	x_3	Slack
$2x_1 + x_2 \leq 8$	Functional	x_4	Slack
$x_2 \leq 3$	Functional	x_5	Slack
$x_1 \geq 0$	Nonnegativity	x_1	Original
$x_2 \geq 0$	Nonnegativity	x_2	Original

(b) Omitted.

(c) As x^4 lies on the intersection of $2x_1 + x_2 = 8$ and $x_2 = 3$, it is $x^4 = (\frac{5}{2}, 3)$. Now, if we plug in x^4 into $2x_1 + x_2 + x_4 = 8$, we get $2 \times \frac{5}{2} + 3 + x_4 = 8$, which means $x_4 = 0$. Similarly, if we plug in x^5 into $x_2 + x_5 = 3$, we get $3 + x_5 = 3$, which means $x_5 = 0$.

(d) The connection between basic feasible solutions and extreme points are shown in the table below. For each extreme point, two variables are chosen to be nonbasic and thus 0. It can be verified that this fits the figure.

x_1	x_2	x_3	x_4	x_5	Extreme point
0	0	4	8	3	x^1
0	-4	0	12	7	(Infeasible)
0	8	12	0	-5	(Infeasible)
0	3	7	5	0	x^5
2	0	0	4	3	x^2
4	0	-4	0	3	(Infeasible)
N/S	0	N/S	N/S	0	(Infeasible)
3	2	0	0	1	x^3
$\frac{7}{2}$	3	0	-2	0	(Infeasible)
$\frac{5}{2}$	3	2	0	0	x^4

(e) For x^4, the basis is $\{x_1, x_2, x_3\}$. For x^5, the basis is $\{x_2, x_3, x_4\}$. As they share two variables, they are adjacent. For x^2, the basis is $\{x_1, x_4, x_5$. As this basis shares only one variable with that of x^4, x^2 and x^4 are not adjacent.

(f) At x^3, the basis is $\{x_1, x_2, x_5\}$. At x^4, the basis is $\{x_1, x_2, x_3\}$. Therefore, from x^3 to x^4, we should let x_5 leave and x_3 enter. At x^5, the basis is $\{x_2, x_3, x_4\}$. Therefore, from x^5 to x^4, we should let x_4 leave and x_1 enter.