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Suggested Solution for Homework 09

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1. For this problem, the demand rate is $D = 48000$ gallons per year, ordering cost is $K = 50$ dollars per order, and holding cost is $h = 0.3$ dollars per gallon per year.

   (a) The optimal order size, which is the EOQ, which be $q* = \sqrt{\frac{2KD}{h}} = 4000$ gallons per order.

   (b) In average, there should be $\frac{D}{q^*} = 12$ orders in a year.

   (c) The order cycle time is $\frac{1}{12} \approx 0.0833$ years, or $0.0883 \times 52 \approx 4.33$ weeks.

   (d) If the lead time is $L = 2$ weeks, as it is shorter than the order cycle time, the reorder point is $R = LD = 1846.15$ gallons.

   (e) If the lead time is $L' = 10$ weeks, as it is longer than the order cycle time, we need to adjust it by subtracting 2 cycle times so that $L' - 2T^* \approx 1.33 < T^*$. The reorder point is then calculated as $(L' - 2T^*)D \approx 1230.77$ gallons.

2. The three cost curves are depicted in Figure 1. As the thick vertical line indicates, the ordering and holding cost curves intersect at the EOQ $q^* = 4000$ gallons.

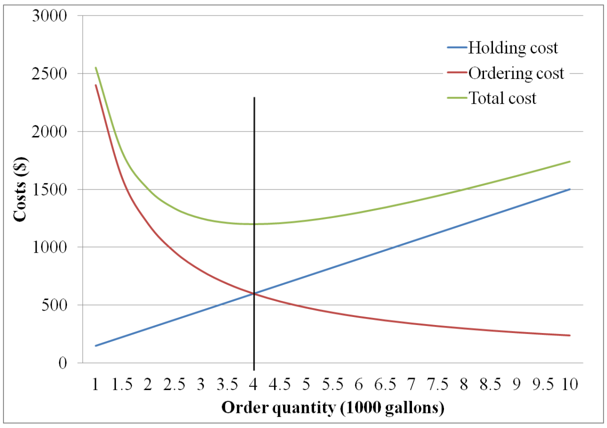

Figure 1: Cost curves for Problem 2

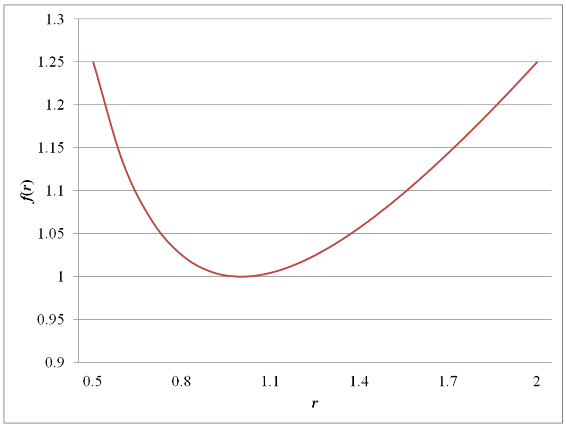

Figure 2: $f(r)$ for Problem 4

3. (a) The total cost under $q^* + d$ is

$$TC(q^* + d) = \frac{h(q^* + d)}{2} + \frac{KD}{q^* + d}.$$

   (b) The total cost under $q^* - d$ is

$$TC(q^* - d) = \frac{h(q^* - d)}{2} + \frac{KD}{q^* - d}.$$

   (c) To see this, we calculate

$$\begin{aligned} TC(q^* - d) - TC(q^* + d) &= -hd + KD\left(\frac{1}{q^* - d} - \frac{1}{q^* + d}\right) \\ &= -hd + KD\left[\frac{2d}{(q^*)^2 - d^2}\right] = d\left[-h + KD\left(\frac{2}{\frac{2KD}{h} - d^2}\right)\right] \\ &= dh\left(-1 + \frac{2KD}{2KD - hd^2}\right) = dh\left(\frac{hd^2}{2KD - hd^2}\right). \end{aligned}$$

As $d > 0$ and $h > 0$, the sign of this term depends on the sign of $2KD - hd^2$. Because

$$d < q^* \Leftrightarrow d^2 < \frac{2KD}{h} \Leftrightarrow hd^2 < 2KD,$$

we know this term is positive and thus $TC(q^* - d) > TC(q^* + d)$ for all $d \in (0, q^*)$.

4. (a) We have

$$f(r) = \frac{TC(rq^*)}{TC(q^*)} = \frac{\frac{hrq^*}{2} + \frac{KD}{rq^*}}{\frac{hq^*}{2} + \frac{KD}{q^*}} = \frac{r\sqrt{\frac{hKD}{2}} + (\frac{1}{r})\sqrt{\frac{hKD}{2}}}{\sqrt{\frac{hKD}{2}} + \sqrt{\frac{hKD}{2}}} = \frac{1}{2}\left(\frac{1}{r} + r\right).$$

(b) The function $f(r)$ over $r \in [\frac{1}{2}, 2]$ is depicted in Figure 2. In particular, $f(\frac{1}{2}) = f(2) = 1.25$.

5. For this problem, the demand rate is $D = 2000 \times 12 = 24000$ units per year, production rate is $r = 100 \times 360 = 36000$ units per year, ordering cost is $K = 1000$ dollars per order, and holding cost is $h = 300$ dollars per unit per year. The effective holding cost is $h' = h(1 - \frac{D}{r}) = 100$ dollars per unit per year.

(a) The optimal order size, which is the EOQ, which be $q^* = \sqrt{\frac{2KD}{h'}} \approx 692.82$ units per order.

(b) In average, there should be $\frac{D}{q^*} \approx 34.64$ orders in a year.

(c) The order cycle time is $\frac{1}{34.64} \approx 0.0289$ years, or $0.0289 \times 12 \approx 0.347$ months.

(d) In a cycle, the slope of the inventory level curve is $r - D$ during production time and $-D$ when there is no production. Simple derivation shows that the proportion of a cycle that is under production is $\frac{D}{r}$. For this problem, we have $\frac{D}{r} = \frac{2}{3}$.

(e) In a cycle, the proportion of a cycle with no production is $1 - \frac{D}{r} = \frac{1}{3}$.

6. For this problem, the overage cost is $c_o = 10$ dollars and the underage cost is $c_u = 25 - 10 = 15$ dollars. Let $D \sim \text{ND}(100, 30)$ be the demand for Christmas tree. The optimal order quantity q^* satisfies

$$\Pr(D < q^*) = \frac{c_u}{c_u + c_o} = 0.6 \quad \Rightarrow \quad \Pr\left(Z < \frac{q^* - 100}{30}\right) = 0.6,$$

where $Z \sim \text{ND}(0, 1)$ is the standard normal random variable. By using a standard normal probability table or any statistical software (such as MS Excel), we get $\frac{q^* - 100}{30} \approx 0.2533$ and thus $q^* \approx 30 \times 0.2533 + 100 \approx 107.6$ units.

7. For this problem, the overage cost is $c_o = 1.2 - 1 = 0.2$ dollars and the underage cost is $c_u = 3 - 1.2 = 1.8$ dollars. Let $D \sim \text{ND}(40, 10)$ be the daily demand for hot dogs and $Z \sim \text{ND}(0, 1)$ is the standard normal random variable.

(a) We have

$$\Pr(D < 52) = \Pr\left(Z < \frac{52 - 40}{10}\right) = \Pr(Z < 1.2) \approx 0.8849,$$

where the last equality can be established with a standard normal probability table or any statistical software (such as MS Excel). Our result shows that the probability that the one-day demands can all be satisfied is around 88.49%.

(b) The optimal order quantity q^* satisfies

$$\Pr(D < q^*) = \frac{c_u}{c_u + c_o} = 0.9 \quad \Rightarrow \quad \Pr\left(Z < \frac{q^* - 40}{10}\right) = 0.9,$$

By using a standard normal probability table or any statistical software (such as MS Excel), we get $\frac{q^* - 40}{10} \approx 1.2816$ and thus $q^* \approx 10 \times 1.2816 + 40 \approx 52.82$ units.

8. (a) Omitted.

   (b) We have

$$\mathbb{E}\big[\min\{D,q\}\big] = \int_0^1 \min\{x,q\}f(x)dx = \int_0^1 \min\{x,q\}dx$$
$$= \int_0^q xdx + \int_q^1 qdx = \frac{1}{2}q^2 + q(1-q).$$

   (c) Let $f(q) = r\mathbb{E}\big[\min\{D,q\}\big] - cq$, we have

$$f(q) = r\left(\frac{1}{2}q^2 + q - q^2\right) - cq = -\frac{r}{2}q^2 + (r-c)q.$$

   This then allows us to calculate the derivatives of $f(q)$:

$$f'(q) = -rq + r - c \quad \text{and} \quad f''(q) = -r < 0.$$

   Therefore, the objective function is to maximize a concave function. As the feasible region is convex, this is a convex program.

   (d) The FOC condition solves the program derived in Part (c). Let q^* be the optimal order quantity, we have

$$-rq^* + r - c = 0 \Rightarrow q^* = \frac{r-c}{r}.$$

   For this problem, $c_u = r-c$, $c_o = c$, and $\frac{c_u}{c_o+c_u} = \frac{r-c}{r}$. The desired equality is then established by recognizing that $F(x) = x$ if F is the cdf of a uniform random variable between 0 and 1.