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Suggested Solution for Homework 10

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1. (a) When $b = 2$, the payoff matrix becomes

	Denial	Confession
Denial	$-1, -1$	$-7, -2$
Confession	$-2, -7$	$-6, -6$

In this case, there are two Nash equilibria: (Denial, Denial) and (Confession, Confession).

- (Denial, Denial) is a Nash equilibrium because no player will deviate ($-2 < -1$).
- (Denial, Confession) is not a Nash equilibrium because player 1 will deviate ($-6 > -7$).
- (Confession, Denial) is not a Nash equilibrium because player 2 will deviate ($-6 > -7$).
- (Confession, Confession) is a Nash equilibrium because no player will deviate ($-7 < -6$).

(b) Suppose $b = 4$. Find all the Nash equilibria, if any. For each Nash equilibrium you find, you need to explain why it is a Nash equilibrium.

When $b = 4$, the payoff matrix becomes

	Denial	Confession
Denial	$-1, -1$	$-5, -4$
Confession	$-4, -5$	$-6, -6$

In this case, there is a unique Nash equilibrium (Denial, Denial).

- (Denial, Denial) is a Nash equilibrium because no player will deviate ($-4 < -1$);
- (Denial, Confession) is not a Nash equilibrium because player 2 will deviate ($-1 > -4$);
- (Confession, Denial) is not a Nash equilibrium because player 1 will deviate ($-1 > -4$);
- (Confession, Confession) is not a Nash equilibrium because both players will deviate ($-5 > -6$).

2. (a) Let a_2 be player 2's action, player 1's best response function is

$$f_1(a_2) = \begin{cases} M & \text{if } a_2 = L \\ T & \text{if } a_2 = C \\ \{T, B\} & \text{if } a_2 = R \end{cases} .$$

(b) Let a_1 be player 1's action, player 2's best response function is

$$f_2(a_1) = \begin{cases} L & \text{if } a_1 = T \\ \{L, C\} & \text{if } a_2 = M \\ R & \text{if } a_3 = B \end{cases} .$$

(c) No player has a strictly dominated strategy.

(d) We need to find intersections of the two best response functions. Below we mark player 1's best response function in bold face and player 2's best response function in italic. It is clear that both (T, C) and (B, R) are Nash equilibria.

	L	C	R
T	$1, \mathit{2}$	$\mathbf{2}, \mathit{1}$	$\mathbf{1}, 0$
M	$\mathbf{2}, 1$	$0, \mathit{1}$	$0, 0$
B	$0, 1$	$0, 0$	$\mathbf{1}, \mathit{2}$

3. (a) The payoff matrix for the two players is

	0	1
0	$0,0$	$3,2$
1	$2,3$	$2,2$

It is clear that both $(0,1)$ and $(1,0)$ are Nash equilibria. $(0,0)$ is not a Nash equilibrium because both players will want to deviate to build the library. $(1,1)$ is also not a Nash equilibrium because both players will want to deviate to free ride.

(b) The payoff marix for the three players are

	0		1	
	0	1	0	1
0	$0,0,0$	$0,-1,0$	$0,0,-1$	$3,2,2$
1	$-1,0,0$	$2,2,3$	$2,3,2$	$2,2,2$

(c) The four Nash equilibria are $(0,0,0)$, $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$. When no player contributes no player will want to waste her one dollar. Moreover, when two players contribute, they will keep their contributions for the library and the free rider will not waste her one dollar. None of $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ is a Nash equilibrium because those who does not contribute will want to contribute. $(1,1,1)$ is also not a Nash equilibrium because all players will want to free ride.

(d) Suppose $k=1$, then for all Nash equilibria we have $x=k=1$: If $x=0$, each player will want to contribute to build the library; if $x \geq 2$, those who are contributing will want to free ride. Suppose $k>1$, then for all Nash equilibria we have $x=k$ or $x \leq k-2$: If $x=k-1$, those who are not contributing will want to contribute to build the library; if $x \geq k+1$, those who are contributing will want to free ride.

4. (a) Suppose (q_1^*, q_2^*, q_3^*) is a Nash equilibrium, we know for player 1, q_1^* solves

$$\max_{q_1 \geq 0} \quad q_1(a-q_1-q_2^*-q_3^*-c),$$

which implies $q_1^* = \frac{a-q_2^*-q_3^*-c}{2}$ or $2q_1^*+q_2^*+q_3^* = a-c$. Similarly, we may find another two equations from players 2's and 3's optimization problems. Collectively, we know the Nash equilibrium must satisfy

$$\begin{array}{ccccccc} 2q_1^* & + & q_2^* & + & q_3^* & = & a-c \\ q_1^* & + & 2q_2^* & + & q_3^* & = & a-c \\ q_1^* & + & q_2^* & + & 2q_3^* & = & a-c. \end{array}$$

Summing all the equations together, we obtain $4(q_1^*+q_2^*+q_3^*) = 3(a-c)$. Using this to subtract each equation above, we get $q_i^* = \frac{a-c}{4}$ for $i=1,...,3$. These then imply the aggregate supply is $Q_3^* = \frac{3(a-c)}{4}$ and the equilibrium price is $P_3^* = \frac{a+3c}{4}$. Recall that when there are two firms, the aggregate supply is $\frac{2(a-c)}{3}$ and the equilibrium price is $\frac{a+2c}{3}$. It is clear that when the number of firms goes from two to three, the aggregate supply increases and the equilibrium price decreases.

(b) Suppose $(q_1^*, ..., q_n^*)$ is a Nash equilibrium, following the same argument as in Part (a), we have

$$q_1^* + q_2^* + \cdots + q_{i-1}^* + 2q_i^* + q_{i+1}^* + \cdots + q_n^* = a-c$$

for $i = 1,2,...,n$. This then results in the individual supply

$$q_i^* = \frac{a-c}{n+1} \quad \forall i = 1,...,n$$

and the aggregate supply and equilibrium price

$$Q_n^* = \frac{n(a-c)}{n+1} \quad \text{and} \quad P_n^* = \frac{a+nc}{n+1}.$$

(c) When n approaches infinity, we have

$$\lim_{n\to\infty} Q_n^* = a - c \quad \text{and} \quad \lim_{n\to\infty} P_n^* = c.$$

In other words, the equilibrium price approaches the marginal cost and the profit margin of each firm approaches to 0.

5. (a) We first try to find player 1's best response by solving

$$\max_{q_1 \geq 0} \quad q_1(a - q_1 - q_2 - c_1).$$

The optimal solution, which is player 1's best response, is $f_1(q_2) = \frac{1-q_2-c_1}{2}$ for $q_2 \in [0, a-c_1]$ and 0 otherwise. Similarly, we may find player 2's best response as $f_2(q_1) = \frac{1-q_1-c_2}{2}$ for $q_1 \in [0, a-c_2]$ and 0 otherwise. Solving $q_1 = \frac{1-q_2-c_1}{2}$ and $q_2 = \frac{1-q_1-c_2}{2}$, we obtain

$$q_1^* = \frac{a + c_2 - 2c_1}{3} \quad \text{and} \quad q_2^* = \frac{a + c_1 - 2c_2}{3}.$$

Given our condition $0 < c_i < \frac{a}{2}$ for $i = 1, 2$, we have $q_1^* \in [0, a - c_2]$ and $q_2^* \in [0, a - c_1]$, which verify that (q_1^*, q_2^*) is indeed a Nash equilibrium.

(b) In this case, we have $f_1(q_2) = \frac{8-q_2}{2}$ for $q_2 \in [0, 8]$ and

$$f_2(q_1) = \begin{cases} \frac{2-q_1}{2} & \text{if } q_1 \in [0,2] \\ 0 & \text{if } q_1 \in [2,8] \end{cases}.$$

The two best response functions are depicted in Figure 1. The Nash equilibrium, as indicated in the figure, is $(q_1^* = 4, q_2^* = 0)$.

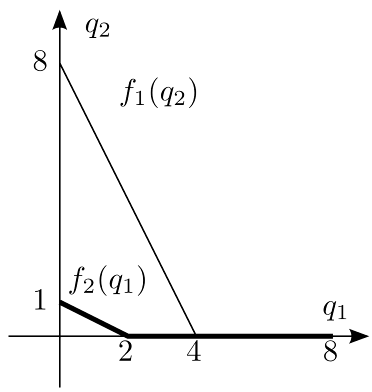

Figure 1: Best response functions for Problem 5b.

(c) In this case, player 2 will have no incentive to produce because player 1's supply will be too high for player 2 to obtain a positive profit margin. Please note that the setting in Part (b) provides an example. The unique Nash equilibrium is $(q_1^* = \frac{1-c_1}{2}, q_2^* = 0)$.